A simple example of the QR Decomosition

Problem (a) Find the QR decomposition of

$$A = \begin{bmatrix} \frac{3}{5} & -\frac{6}{5} \\ \frac{4}{5} & \frac{17}{5} \end{bmatrix}$$

(b) Verify that indeed $A = QR$.

Solution to (a) : First we must find an orthonormal basis for the column space of A. The input is

$$\mathbf{a_1} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad , \quad \mathbf{a_2} = \begin{bmatrix} -\frac{6}{5} \\ \frac{17}{5} \end{bmatrix} \quad .$$

Now

$$\mathbf{v_1} = \mathbf{a_1} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad ,$$

$$\mathbf{v_2} = \mathbf{a_2} - (\frac{\mathbf{a_2}.\mathbf{v_1}}{||\mathbf{v_1}||^2})\mathbf{v_1}$$

We have

$$\mathbf{a_2}.\mathbf{v_1}$$

$$= \begin{bmatrix} -\frac{6}{5} \\ \frac{17}{5} \end{bmatrix} . \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

$$= (-\frac{6}{5})(\frac{3}{5}) + (\frac{17}{5})(\frac{4}{5}) = \frac{50}{25} = 2$$

$$||\mathbf{v_1}||^2 = (\frac{3}{5})^2 + (\frac{4}{5})^2 = 1 \quad .$$

Hence

$$\mathbf{v_2} = \mathbf{a_2} - \frac{2}{1}\mathbf{v_1} = \begin{bmatrix} -\frac{6}{5} \\ \frac{17}{5} \end{bmatrix} - 2 \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{5} \\ \frac{17}{5} \end{bmatrix} - \begin{bmatrix} \frac{6}{5} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ \frac{9}{5} \end{bmatrix}$$

Hence, an **orthogonal** set with the same span is

$$\mathbf{v_1} = \begin{bmatrix} \frac{3}{5} \\ \\ \frac{4}{5} \end{bmatrix} \quad , \quad \mathbf{v_2} = \begin{bmatrix} -\frac{12}{5} \\ \\ \frac{9}{5} \end{bmatrix} \quad .$$

Next we have to find an **orthonormal** basis, by **normailizing** $\mathbf{v_1}$ and $\mathbf{v_2}$.

$$||\mathbf{v_1}|| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = 1$$

$$||\mathbf{v_2}|| = \sqrt{(-\frac{12}{5})^2 + (\frac{9}{5})^2} = \sqrt{\frac{144+81}{5^2}} = \sqrt{\frac{225}{5^2}} = \sqrt{\frac{15^2}{5^2}} = 3 \quad .$$

Hence

$$\mathbf{w_1} = \frac{1}{||\mathbf{v_1}||}\mathbf{v_1} = \frac{1}{1}\begin{bmatrix} \frac{3}{5} \\ \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \\ \frac{4}{5} \end{bmatrix} \quad ,$$

$$\mathbf{w_2} = \frac{1}{||\mathbf{v_2}||}\mathbf{v_2} = \frac{1}{3}\begin{bmatrix} -\frac{12}{5} \\ \\ \frac{9}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \\ \frac{3}{5} \end{bmatrix} \quad .$$

So the output for Gram-Schmidt, after the **normalizations** is:

$$\mathbf{w_1} = \begin{bmatrix} \frac{3}{5} \\ \\ \frac{4}{5} \end{bmatrix} \quad , \quad \mathbf{w_2} = \begin{bmatrix} -\frac{4}{5} \\ \\ \frac{3}{5} \end{bmatrix} \quad .$$

To get Q we just form the matrix whose two columns are $\mathbf{w_1}$ and $\mathbf{w_2}$:

$$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \quad .$$

Now R is the upper triangular matrix

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \quad ,$$

where

$$r_{11} = \mathbf{w_1}.\mathbf{a_1} \quad , \quad r_{12} = \mathbf{w_1}.\mathbf{a_2} \quad , \quad r_{22} = \mathbf{w_2}.\mathbf{a_2} \quad .$$

We have

$$r_{11} = (\frac{3}{5})(\frac{3}{5}) + (\frac{4}{5})(\frac{4}{5}) = 1$$

$$r_{12} = (\frac{3}{5})(-\frac{6}{5}) + (\frac{4}{5})(\frac{17}{5}) = \frac{-18+68}{25} = \frac{50}{25} = 2 \quad .$$

$$r_{22} = (-\frac{4}{5})(-\frac{6}{5}) + (\frac{3}{5})(\frac{17}{5}) = \frac{24+51}{25} = \frac{75}{25} = 3 \quad .$$

Hence

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad .$$

Answer to the problem: The QR decomposition of A is

$$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \quad . \quad , \quad R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad .$$

Solution to (b)

$$QR = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad .$$

$$= \begin{bmatrix} (\frac{3}{5})(1) + (-\frac{4}{5})(0) & (\frac{3}{5})(2) + (-\frac{4}{5})(3) \\ (\frac{4}{5})(1) + (-\frac{3}{5})(0) & (\frac{4}{5})(2) + (\frac{3}{5})(3) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & -\frac{6}{5} \\ \frac{4}{5} & \frac{17}{5} \end{bmatrix} \quad .$$

Yea! It equals A.