Solution to Problem 26 in Section 6.2 of "Elementary Linear Algebra" by Spence, Insel, and Friedberg

Problem (a) Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 0 \end{bmatrix}$$

(b) Verify that indeed $A = QR$.

Solution: First we must find an orthonormal basis for the column space of A. The input is

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad , \quad \mathbf{a_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad .$$

Now

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad ,$$

$$\mathbf{v_2} = \mathbf{a_2} - (\frac{\mathbf{a_2}.\mathbf{v_1}}{||\mathbf{v_1}||^2})\mathbf{v_1}$$

We have

$$\mathbf{a_2}.\mathbf{v_1}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} . \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= (1)(1) + (-1)(-2) + (0)(1) = 3$$

$$||\mathbf{v_1}||^2 = (1)^2 + (-2)^2 + 1^2 = 6 \quad .$$

Hence

$$\mathbf{v_2} = \mathbf{a_2} - \frac{3}{6}\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \quad .$$

Hence, an orthogonal set with the same span is

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v_2} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \quad .$$

Next we have to find an **orthonormal** basis, by **normailizing** $\mathbf{v_1}$ and $\mathbf{v_2}$.

$$||\mathbf{v_1}|| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6} \quad .$$

$$||\mathbf{v_2}|| = \sqrt{(1/2)^2 + (0)^2 + (-1/2)^2} = \frac{1}{\sqrt{2}} \quad .$$

Hence

$$\mathbf{w_1} = \frac{1}{||\mathbf{v_1}||}\mathbf{v_1} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{2\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \quad .$$

$$\mathbf{w_2} = \frac{1}{||\mathbf{v_2}||}\mathbf{v_2} = \sqrt{2} \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad .$$

So the output for Gram-Schmidt, after the normalizations is:

$$\mathbf{w_1} = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{2\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \quad , \quad \mathbf{w_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad .$$

To get Q we just form the matrix whose two columns are $\mathbf{w_1}$ and $\mathbf{w_2}$:

$$Q = \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\ -\frac{2\sqrt{6}}{6} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \end{bmatrix} \quad , \quad .$$

Now R is the upper triangular matrix

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \quad ,$$

where

$$r_{11} = \mathbf{w_1}.\mathbf{a_1} \quad , \quad r_{12} = \mathbf{w_1}.\mathbf{a_2} \quad , \quad r_{22} = \mathbf{w_2}.\mathbf{a_2} \quad .$$

We have

$$r_{11} = (\frac{\sqrt{6}}{6})(1) + (-\frac{2\sqrt{6}}{6})(-2) + (\frac{\sqrt{6}}{6})(1) = \sqrt{6}(\frac{1}{6} + \frac{2}{3} + \frac{1}{6}) = \sqrt{6}$$

$$r_{12} = (\frac{\sqrt{6}}{6})(1) + (-\frac{2\sqrt{6}}{6})(-1) + (\frac{\sqrt{6}}{6})(0) = \sqrt{6}(\frac{1}{6} + \frac{1}{3}) = \frac{\sqrt{6}}{2}$$

$$r_{22} = (\frac{\sqrt{2}}{2})(1) + (0)(-1) + (-\frac{\sqrt{2}}{2})(0) = \frac{\sqrt{2}}{2}$$

Hence

$$R = \begin{bmatrix} \sqrt{6} & \frac{\sqrt{6}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} .$$

Answer to the problem: The QR decomposition of A is

$$Q = \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\ -\frac{2\sqrt{6}}{6} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \end{bmatrix} , \quad R = \begin{bmatrix} \sqrt{6} & \frac{\sqrt{6}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} .$$

You are welcome to check that indeed $QR = A$.