Solutions to the Attendance Quiz for Lecture 18

1. Find the eigenvues of the following matrix, and determine a basis for each eigenspace.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 9 & 3 & 10 \\ -5 & 0 & -2 \end{bmatrix} \quad .$$

Sol. of 1:

Part 1: Find the eigenvalues, with their multiplicitcy. We form $A - tI_3$ by subtracting t from the **diagonal entries** of A:

$$A - tI_3 = \begin{bmatrix} 3-t & 0 & 0 \\ 9 & 3-t & 10 \\ -5 & 0 & -2-t \end{bmatrix} \quad .$$

Using co-factor expansion with respect to the **first** row:

$$\det(A - tI_3) = \det \begin{bmatrix} 3-t & 0 & 0 \\ 9 & 3-t & 10 \\ -5 & 0 & -2-t \end{bmatrix}$$

$$= (3-t)\det \begin{bmatrix} 3-t & 10 \\ 0 & -2-t \end{bmatrix} + 0 + 0 = (3-t)((3-t)(-2-t) - (10)(0)) = (t-3)^2(t+2) \quad .$$

So the **characteristic polynomial** is $(t-3)^2(t+2)$ and the **characteristic equation** is

$$(t-3)^2(t+2) = 0 \quad .$$

Solving the charactestic equation we get the eigenvalues:

$t = 3$ (multiplicity 2)

$t = -2$ (multiplicity 1).

Comments: About %85 people did this first part correctly.

Part 2: For each of these eigenvalues, we find a basis to their **eigenspace**.

For $t = 3$ we get:

$$A - 3I_3 = \begin{bmatrix} 3-3 & 0 & 0 \\ 9 & 3-3 & 10 \\ -5 & 0 & -2-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 9 & 0 & 10 \\ -5 & 0 & -5 \end{bmatrix} \quad .$$

We have to find the **null space** of this matrix. In everyday notation:

$$\begin{bmatrix} 0 & 0 & 0 \\ 9 & 0 & 10 \\ -5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad .$$

Now we can do it either by **high school algebra** (alas, some people forgot and messed up!) or the systematic way, with elementary row operations. Let's do it the high-school way.

In everyday notation we have the system

$$0 = 0$$

$$9x_1 + 10x_3 = 0$$

$$-5x_1 - 5x_3 = 0$$

From the second equation we get $x_1 = -\frac{10}{9}x_3$, From the third equation we get $x_1 = -x_3$, setting these equal to each other we get $x_3 = 0$, so also $x_1 = 0$ and x_2 could be what it wants to be. So the general solution is:

$$x_1 = 0$$

$$x_2 = x_2$$

$$x_3 = 0$$

In vector notation this is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So a basis for the eigenspace for $t = 3$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Comment: You get the same answer if you use Gaussian elimination. If you are not comfortable using high-school methods, use Gaussian elimination.

Next we have to worry about For $t = -2$ we get:

$$A - (-2)I_3 = \begin{bmatrix} 3+2 & 0 & 0 \\ 9 & 3+2 & 10 \\ -5 & 0 & -2+2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 9 & 5 & 10 \\ -5 & 0 & 0 \end{bmatrix} .$$

For the sake of variety, let's find the null space of this matrix using elementary row operations.

$$\begin{bmatrix} 5 & 0 & 0 \\ 9 & 5 & 10 \\ -5 & 0 & 0 \end{bmatrix} \begin{matrix} (1/5)r_1 \to r_1 \\ \longrightarrow \\ (-1/5)r_3 \to r_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 9 & 5 & 10 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} r_2 - 9r_1 \to r_2 \\ \longrightarrow \\ r_3 - r_1 \to r_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 10 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} (1/5)r_2 \to r_2 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} .$$

This is in **reduced-row-echelon form**. In everyday notation this is:

$$x_1 = 0$$

$$x_2 + 2x_3 = 0$$

$$0 = 0 \quad .$$

x_3 is a **free variable**, and the general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \quad .$$

So a basis for the eigenspace for $t = -2$ is $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Final answer:

The eigenvalues with the correponding bases for the eigenspaces are:

$t = 3$ (multiplicity 2), that has an eigenspace whose basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$t = -2$ (multiplicity 1), that has an eigenspace whose basis is $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Comments: Only about one third of the people got it completely. Quite a few people messed up finding the eignespaces. If you are not comfortable using high-school methods, use Gaussian elimination.