Solutions to Dr. Z.'s Math 250(2), (Fall 2010, RU) REAL Quiz #9 (Dec. 2, 2010)

1. (4 points) A matrix A is given. Find, if possible, an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Otherwise explain why A is not diagonalizable.

$$A = \begin{bmatrix} -8 & 6 \\ -15 & 11 \end{bmatrix} \quad ,$$

Sol. to 1:

$$\det(A - tI_2) = \det \begin{bmatrix} -8-t & 6 \\ -15 & 11-t \end{bmatrix} = (-8-t)(11-t) - 6(-15) = (t+8)(t-11) + 90 = t^2 - 3t - 88 + 90 = t^2 - 3t + 2 \quad .$$

To find the **eigenvalues** we need to solve the **characteristic equation**: $t^2 - 3t + 2 = 0$. Factoring, we get $(t-1)(t-2) = 0$ so the eigenvalues are $t = 1$ and $t = 2$, and we have found D

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad .$$

To find P we need to find **eigenvectors** corresponding to each of these eigenvalues.

When $t = 1$ we have to solve

$$(A - (1)I_2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad .$$

This means:

$$\begin{bmatrix} -8-1 & 6 \\ -15 & 11-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad .$$

This means

$$\begin{bmatrix} -9 & 6 \\ -15 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad .$$

In everyday notation, we have to solve the system

$$-9x_1 + 6x_2 = 0 \quad ,$$

$$-15x_1 + 10x_2 = 0 \quad .$$

Not surprisingly, the second equation is a multiple of the first, so we only have to consider the first. taking x_2 as the free variable we have that the general solution is

$$x_1 = \frac{6}{9}x_2 = \frac{2}{3}x_2 \quad ,$$

$$x_2 = x_2 \quad .$$

In **vector notation**, this is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

So an eigenvector corresponding to the eigenvalue $t = 1$ is: $\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$. Since we are allowed to multiply by any non-zero number, let's clear the fraction and multiply by 3 getting: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. This is the **first** column of P.

When $t = 2$ we have to solve

$$(A - 2I_2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad .$$

This means:

$$\begin{bmatrix} -8-2 & 6 \\ -15 & 11-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad .$$

This means

$$\begin{bmatrix} -10 & 6 \\ -15 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad .$$

In everyday notation, we have to solve the system

$$-10x_1 + 6x_2 = 0 \quad ,$$

$$-15x_1 + 9x_2 = 0 \quad .$$

Not surprisingly, the second equation is a multiple of the first, so we only have to consider the first. taking x_2 as the free variable we have that the general solution is

$$x_1 = \frac{6}{10}x_2 = \frac{3}{5}x_2 \quad ,$$

$$x_2 = x_2 \quad .$$

In **vector notation**, this is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \quad .$$

So an eigenvector corresponding to the eigenvalue $t = 2$ is: $\begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}$. Since we are allowed to multiply by any non-zero number, let's clear the fraction and multiply by 5 getting: $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$. This is the **second** column of P.

Ans. to 1:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad , \quad P = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \quad .$$

2. (4 points) Use **1** to compute A^6 (No credit for other methods!)

Reminder 1: $2^6 = 64$

Reminder 2 The quickest way to find the inverse of a 2×2 matrix is to use the formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad .$$

Sol. to 2: We first need to compute P^{-1}. Using Reminder 2, we have

$$P^{-1} = \frac{1}{(2)(5) - (3)(3)} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \quad .$$

Now $A^6 = PD^6P^{-1}$.

D^6 is easy:

$$D^6 = \begin{bmatrix} 1^6 & 0 \\ 0 & 2^6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 64 \end{bmatrix} \quad .$$

So:

$$A^6 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 64 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -192 & 128 \end{bmatrix} = \begin{bmatrix} -566 & 378 \\ -945 & 631 \end{bmatrix} \quad .$$

This is the **ans.**

3. (1 point each). True or False. Explain briefly!

(a) If the characteristic polynomial of a matrix A factors into a product of linear factors, then A is diagonalizable.

Sol. to 3a): **False**. They may be multiple roots, in that case it is sometimes not diagonalizable (whenever there is an eigenvalue whose dimension of corresponding eigenspace is **less** than the multiplicity. The corrected statement is: "... product of **distinct** linear factors".

(b) If A is a diagonalizable matrix, then there is a unique diagonal matrix D such that $A = PDP^{-1}$.

Sol. of 3b): **False**. You can order the eigenvalues any way you want, as long as you place them on the diagonal of D and make all the other entries 0.