1. (5 points) Find a basis for the null space of the matrix

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & -1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\]

**Sol. of 1:** We find the reduced row echelon form of the matrix:

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & -1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 0 \\
0 & -1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

In everyday notation the system \( Rx = 0 \) is

\[
\begin{align*}
x_1 + 2x_3 &= 0 \\
x_2 - x_3 &= 0 \\
0 &= 0
\end{align*}
\]

Solving we get:

\[
\begin{align*}
x_1 &= -2x_3 \\
x_2 &= x_3 \\
x_3 &= x_3 \quad (\text{free})
\end{align*}
\]

In vector notation this is:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-2x_3 \\
x_3 \\
x_3
\end{bmatrix} = x_3 \begin{bmatrix}
-2 \\
1 \\
1
\end{bmatrix}
\]

Since there is only one free variable \((x_3)\), the dimension of the null space (alias the nullity of \( A \)) is 1 and a basis is:

\[
\left\{ \begin{bmatrix}
-2 \\
1 \\
1
\end{bmatrix} \right\}
\]

2. (5 points, 1 each) Determine whether the following statements are True or False. Give a short explanation (or an example or counter-example) in each case. (**No credit without explanation**).

(a) The subspace \( \{0\} \) is called the null-space.

**Sol. of 2a:** False. It is called the **zero-space**.
(b) If $S$ is a linearly independent subset and $\text{Span } S = V$, then $S$ is a generating set for $V$.

**Sol. of 2b): True.** It sure is. It is enough that $\text{Span } S = V$ to make $S$ a generating set, so the fact that it is linearly independent is irrelevant.

(c) $\mathbb{R}^3$ contains at least three different subspaces.

**Sol. of 2c): True.** 3 is a very gross underestimate! It has infinitely many subspaces.

(d) A vector $v$ is in $\text{Col } A$ if and only if $Ax = v$ is consistent.

**Sol. of 2d): True.** If a vector $v$ is in $\text{Col } A$ it means that it is a **linear combination** of its columns, that means that it can be expressed in the form $Ax$ for some vector of numbers $x$ which means that the system of equations $Ax = v$ has a solution, which means that the system of equations $Ax = v$ is **consistent**.

(e) The pivot columns of the matrix $A$ (i.e. the columns corresponding to the pivot columns of the reduced-row-echelon form $R$, but those of $A$, not of $R$) always form a basis for its column space.

**Sol. of 2e): True.** By the column correspondence property. The pivot columns of $R$ obviously form a basis for its own column space (not to be confused with the column space of $A$ that is (usually) entirely different). By the column correspondence property it follows that the corresponding columns in $A$ form a basis for the column space of $A$. 