Solutions to Dr. Z.'s Math 250(1), (Fall 2010, RU) REAL Quiz #5 (Oct. 21, 2010)

1. a) Find an LU decomposition of the following matrix

$$\begin{bmatrix} 3 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 2 \end{bmatrix}$$

Sol. of 1a): You use the first phase of Gaussian elimination, **but** only using the elementary row operations of the type $r_i + cr_j \to r_i$, until you get an **upper triangular** matrix U, taking careful record of these elementary row operations.

Here goes:

$$\begin{bmatrix} 3 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 2 \end{bmatrix} \overset{r_2-(4/3)r_1 \to r_2, r_3+(2/3)r_1 \to r_3}{\to} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1/3 & 8/3 \\ 0 & 1/3 & 8/3 \end{bmatrix} \overset{r_3-r_2 \to r_3}{\to} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1/3 & 8/3 \\ 0 & 0 & 0 \end{bmatrix}$$

This is U. To get L we form a 3×3 matrix with 1's on the diagonal, 0 above the diagonal, and

$L_{2,1} = 4/3$ (because we used the elementary-row-operation $r_2 - (4/3)r_1 \to r_2$)

$L_{3,1} = -2/3$ (because we used the elementary-row-operation $r_2 + (2/3)r_1 \to r_2$)

$L_{3,2} = 1$ (because we used the elementary-row-operation $r_3 + (-1)r_2 \to r_3$).

So we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ -2/3 & 1 & 1 \end{bmatrix} \quad .$$

Ans. to 1a):

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ -2/3 & 1 & 1 \end{bmatrix} \quad , \quad U = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1/3 & 8/3 \\ 0 & 0 & 0 \end{bmatrix} \quad .$$

b) Use the answer to part a) to solve the following system of linear equations:

$$3x_1 - x_2 + x_3 = -1$$

$$4x_1 - x_2 + 4x_3 = -2$$

$$-2x_1 + x_2 + 2x_3 = -2$$

Sol. of 1b): We have to solve $A\mathbf{x} = \mathbf{b}$ with the matrix A as above. We now know that $A = LU$ so we have to solve $LU\mathbf{x} = \mathbf{b}$. Putting $\mathbf{y} = U\mathbf{x}$, we have $L\mathbf{y} = \mathbf{b}$. So we first have to solve

$$L\mathbf{y} = \mathbf{b} \quad .$$

In everyday notation this is

$$y_1 = -1 \quad ,$$

$$(4/3)y_1 + y_2 = -2 \quad ,$$

$$(-2/3)y_1 + y_2 + y_3 = -2 \quad .$$

We get $y_1 = -1$ for free. Plugging-into the second equation we get $(4/3)(-1) + y_2 = -2$ so $y_2 = -2 + 4/3 = -2/3$ and finally $(-2/3)(-1) + (-2/3) + y_3 = -2$ gives $y_3 = -2$.

Now we are ready to solve $U\mathbf{x} = \mathbf{y}$. (Using the y_1, y_2, y_3 that we have just found). In everyday notation this is:

$$3x_1 - x_2 + x_3 = -1$$

$$(1/3)x_2 + (8/3)x_3 = -2/3$$

$$0 = -2 \quad .$$

But this is **NONSENSE**. The system is inconsistent!

Ans. to 1b): NO SOLUTIONS!