1. a) Find an LU decomposition of the following matrix

\[
\begin{bmatrix}
3 & -1 & 1 \\
4 & -1 & 4 \\
-2 & 1 & 2
\end{bmatrix}
\]

**Sol. of 1a):** You use the first phase of Gaussian elimination, but only using the elementary row operations of the type \( r_i + cr_j \rightarrow r_i \), until you get an upper triangular matrix \( U \), taking careful record of these elementary row operations.

Here goes:

\[
\begin{bmatrix}
3 & -1 & 1 \\
4 & -1 & 4 \\
-2 & 1 & 2
\end{bmatrix}
\]

\[
r_2 - \left(\frac{4}{3}\right)r_1 \rightarrow r_2, r_3 + \left(\frac{2}{3}\right)r_1 \rightarrow r_3
\]

\[
\begin{bmatrix}
3 & -1 & 1 \\
0 & 1/3 & 8/3 \\
0 & 1/3 & 8/3
\end{bmatrix}
\]

\[
r_3 - r_2 \rightarrow r_3
\]

\[
\begin{bmatrix}
3 & -1 & 1 \\
0 & 1/3 & 8/3 \\
0 & 0 & 0
\end{bmatrix}
\]

This is \( U \). To get \( L \) we form a \( 3 \times 3 \) matrix with 1’s on the diagonal, 0 above the diagonal, and

\[ L_{2,1} = \frac{4}{3} \text{ (because we used the elementary-row-operation } r_2 - \left(\frac{4}{3}\right)r_1 \rightarrow r_2) \]

\[ L_{3,1} = -\frac{2}{3} \text{ (because we used the elementary-row-operation } r_2 + \left(\frac{2}{3}\right)r_1 \rightarrow r_2) \]

\[ L_{3,2} = 1 \text{ (because we used the elementary-row-operation } r_3 + (-1)r_2 \rightarrow r_3). \]

So we have

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
4/3 & 1 & 0 \\
-2/3 & 1 & 1
\end{bmatrix}
\]

**Ans. to 1a):**

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
4/3 & 1 & 0 \\
-2/3 & 1 & 1
\end{bmatrix}, \quad U = \begin{bmatrix}
3 & -1 & 1 \\
0 & 1/3 & 8/3 \\
0 & 0 & 0
\end{bmatrix}
\]

b) Use the answer to part a) to solve the following system of linear equations:

\[
3x_1 - x_2 + x_3 = -1
\]

\[
4x_1 - x_2 + 4x_3 = -2
\]

\[
-2x_1 + x_2 + 2x_3 = -2
\]

**Sol. of 1b):** We have to solve \( Ax = b \) with the matrix \( A \) as above. We now know that \( A = LU \) so we have to solve \( LUx = b \). Putting \( y = Ux \), we have \( Ly = b \). So we first have to solve \( Ly = b \) .
In everyday notation this is

\[ y_1 = -1 \]
\[ (4/3)y_1 + y_2 = -2 \]
\[ (-2/3)y_1 + y_2 + y_3 = -2 \]

We get \( y_1 = -1 \) for free. Plugging into the second equation we get \((4/3)(-1) + y_2 = -2\) so \( y_2 = -2 + 4/3 = -2/3 \) and finally \((-2/3)(-1) + (-2/3) + y_3 = -2\) gives \( y_3 = -2 \).

Now we are ready to solve \( Ux = y \). (Using the \( y_1, y_2, y_3 \) that we have just found). In everyday notation this is:

\[ 3x_1 - x_2 + x_3 = -1 \]
\[ (1/3)x_2 + (8/3)x_3 = -2/3 \]
\[ 0 = -2 \]

But this is NONSENSE. The system is inconsistent!

**Ans. to 1b): NO SOLUTIONS!**