Dr. Z.'s Math 250(1), (Fall 2010, RU) SOLUTIONS to the REAL Quiz #2 (Sept. 23, 2010)

1. (4 pts.) Determine whether the given system is consistent, and if so, find its general solution.

$$x_1 - x_2 + x_4 = -4$$

$$x_1 - x_2 + 2x_4 + 2x_5 = -5$$

$$3x_1 - 3x_2 + 2x_4 - 2x_5 = -11 \quad .$$

Sol. of 1: The **augmented** matrix of the system is (with variables x_1, x_2, x_3, x_4, x_5) is

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & -4 \\ 1 & -1 & 0 & 2 & 2 & -5 \\ 3 & -3 & 0 & 2 & -2 & -11 \end{bmatrix} \quad .$$

We must bring it first to **row echelon form**. If we are lucky we'll get a row of 0's except for a non-zero entry at the end, and we would conclude that the system is inconsistent. If we are lucky, and we won't get such a scenario, then we must go on to phase II and get it in **reduced row echelon form**, so that we can find the general solution.

Doing $r_2 - r_1 \rightarrow r_2$ and $r_3 - 3r_1 \rightarrow r_3$ yields

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{bmatrix} \quad .$$

the pivot on the second row is at column 4 (the $(2,4)$ entry) and we want to make 0 everything below it. Doing $r_2 + r_3 \rightarrow r_3$ gives

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad .$$

This is in **row-echelon** form, and since we don't have an all-0-row-followed-by-non-zero (we have a complete 0 row, but that's OK), we conclude that the system is **consistent**, and unfortunately, we must proceed to the next phase.

Doing $r_1 - r_2 \rightarrow r_1$ yields

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad .$$

This is in **reduced-row-echelon form**. In everyday language this means

$$x_1 - x_2 - 2x_5 = -3 \quad ,$$

$$x_4 + 2x_5 = -1 \quad .$$

The **basic variables** (corresponding to the pivot columns 1 and 4) are x_1 and x_4 and the **free variables** are the remaining variables x_2, x_3, x_5. The general solution is

$$x_1 = -3 + x_2 + 2x_5 \quad , \quad x_2 = x_2 \quad , \quad x_3 = x_3 \quad ,$$

$$x_4 = -1 - 2x_5 \quad , \quad x_5 = x_5 \quad .$$

Comments: About %40 of the people got it completely right. Many people knew how to do it but messed up sooner or later (they got lots of partial credit). Quite a few people wrote $x_3 = 0$. This is wrong, $x_3 = x_3$ is a free variable, and can be anything. It is true that it never shows up, but this means that it is **free** to be anything it wants, not that it is 0.

2. (4 pts.) Find a subset of the following set $\mathcal{S}$ of vectors in R^3 with the same span as $\mathcal{S}$ that is as small as possible.

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \quad , \quad \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} \right\} \quad .$$

Sol. of 2: By inspection the third vector is the sum of the first two, so we can **kick-out** the third vector leaving us with the smaller set

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \right\} \quad .$$

Since the two surviving vectors are not multiples of each other, we can't kick-out any more vectors, so this is the answer.

Comment: About %50 of the people got it right. Quite a few people didn't understand what they had to do, and did something completely different (like trying to solve a system of equations).

3. True or False (Explain when appropriate)

(a) (1 pt.) The number of pivot columns of a matrix equals its rank.

Sol. of 3a): True (by definition, pivot columns correspond to pivot entries, and their number is the number of non-zero rows in the reduced row echelon form.)

(b) (1 pt.) Every finite subset of R^n is contained in its span.

Sol. of 3b): True. Every vector of the set can be written as linear combination with coefficient 1 for that vector and coefficient 0 for the other guys.

For example: if $S = \{\mathbf{u}, \mathbf{v}\}$,

$$\mathbf{u} = 1 \cdot \mathbf{u} + 0 \cdot \mathbf{v} \quad ,$$

$$\mathbf{v} = 0 \cdot \mathbf{u} + 1 \cdot \mathbf{v} \quad .$$