Solutions to Dr. Z.'s Math 250(1), (Fall 2010, RU) REAL Quiz #10 (Dec. 9, 2010)

1. (5 points) Apply the Gram-Schmidt process to replace the given linearly independent set $\mathcal{S}$ by an **orthonormal** set with the same span as $\mathcal{S}$.

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad , \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \quad .$$

Sol. of 1:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad , \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Applying Gram-Schmidt:

$$\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{(1)(1) + (-2)(-1) + (1)(0)}{(1)^2 + (-2)^2 + (1)^2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} \quad .$$

Finally

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$\mathbf{w}_2 = \frac{\mathbf{v}_2}{||\mathbf{v}_2||} = \sqrt{2} \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix} \quad .$$

Ans. to 1:

$$\left\{ \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad , \quad \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix} \right\}$$

2.(5 points) Using **1**, find matrices Q and R in a QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 0 \end{bmatrix} \quad .$$

Sol. of 2; Q is easy: it is simply the matrix whose colums are $\mathbf{w}_1, \mathbf{w}_2$:

$$Q = \begin{bmatrix} 1/\sqrt{6} & \sqrt{2}/2 \\ -2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -\sqrt{2}/2 \end{bmatrix} \quad .$$

The **upper-triangular** matrix R is

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

where

$$r_{11} = \mathbf{a}_1 \cdot \mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \frac{6}{\sqrt{6}} = \sqrt{6} \quad ,$$

$$r_{12} = \mathbf{a}_2 \cdot \mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2} \quad ,$$

$$r_{22} = \mathbf{a}_2 \cdot \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix} = \frac{\sqrt{2}}{2} \quad .$$

So

$$R = \begin{bmatrix} \sqrt{6} & \sqrt{6}/2 \\ 0 & \sqrt{2}/2 \end{bmatrix} \quad .$$

Ans. to 2:

$$Q = \begin{bmatrix} 1/\sqrt{6} & \sqrt{2}/2 \\ -2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -\sqrt{2}/2 \end{bmatrix} \quad , \quad R = \begin{bmatrix} \sqrt{6} & \sqrt{6}/2 \\ 0 & \sqrt{2}/2 \end{bmatrix} \quad .$$