1. (a) Find an $LU$ decomposition of the matrix
\[
\begin{bmatrix}
-1 & 2 & 1 & -1 & 3 \\
1 & -4 & 0 & 5 & -5 \\
-2 & 6 & -1 & -5 & 7 \\
-1 & -4 & 4 & 11 & -2
\end{bmatrix}
\]

(b) Use the results of part (a) to solve the system
\[
\begin{align*}
-x_1 + 2x_2 + x_3 - x_4 + 3x_5 &= 7 \\
x_1 - 4x_2 - 5x_4 - 5x_5 &= -7 \\
-2x_1 + 6x_2 - x_3 - 5x_4 + 7x_5 &= 6 \\
x_1 - 4x_2 + 4x_3 + 11x_4 - 2x_5 &= 11
\end{align*}
\]

2. Let $A$ be a $4 \times 4$ matrix with row vectors $a, b, c, d$ that is with
\[
A = \begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
\]
and determinant equal to 5. Find

(a) The determinant of the matrix
\[
\begin{bmatrix}
3a + b + d \\
b \\
c + d \\
d
\end{bmatrix}
\]

(b) The determinant of the matrix $AC^2AC$ where
\[
C = \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 2 & 6 & 11 \\
0 & 0 & 1 & -17 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

3. Compute the determinant by using elementary row operations (no credit for other methods)
\[
\begin{bmatrix}
1 & -1 & 2 \\
2 & -1 & -1 \\
-8 & 10 & -20
\end{bmatrix}
\]
4. (a) Suppose that $A$ and $B$ are $4 \times 5$ matrices and that $B$ is obtained from $A$ by the elementary row operations given below. In each case give an elementary matrix $E$ such that $B = EA$

(i) $r_1 \leftrightarrow r_4$

(ii) $r_3 + 3r_2 \rightarrow r_3$

(b) Give the inverses of the elementary matrices found in (i) and (ii).

5. For what values of $d$ is the given matrix not invertible.

$$
\begin{bmatrix}
-d & 1 & 1 \\
d & -2 & 4 \\
1 & 2 & 3
\end{bmatrix}
$$

6. A certain $3 \times 3$ matrix has reduced row echelon form

$$
R = \begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
$$

Find explicitly a non-trivial linear relation on the columns of $A$, that is a relation $c_1a_1 + c_2a_2 + c_3a_3 = 0$, with $c_1, c_2, c_3$ not all zero.

7. Explain why the following set is a subspace of $\mathbb{R}^5$ and find a basis for it.

$$
\left\{ \begin{bmatrix}
r + s + 2t \\
r - s \\
3r + 2s + 5t \\
-2r + 3s + t \\
r - s - t
\end{bmatrix} : r, s, and t are scalars \right\} \subseteq \mathbb{R}^5
$$

8. Let $v$ be a non-zero vector in $\mathbb{R}^2$, and let $A = vv^T$ ($A$ is a $2 \times 2$ matrix.)

(a) Show that $v$ is an eigenvector of $A$. What is the eigenvalue?

(b) What is the rank of $A$? What is the other eigenvalue of $A$?

9. Explain why the following sets in $\mathbb{R}^4$ are not subspaces

(a)

$$
\left\{ \begin{bmatrix}
s \\
2s \\
3s \\
5s
\end{bmatrix} : s \geq 0 \right\}
$$
(b) 
\[
\left\{ \begin{bmatrix} 1 + t \\ 2 + t \\ 3 + t \\ 10 - t \end{bmatrix} \right\} \in \mathbb{R}^4 : t \text{ is a scalar}
\]

10. The matrix 
\[
A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & -6 & 9 \\ -5 & -1 & 5 \end{bmatrix}
\]
has characteristic polynomial \(- (t + 2)^2(t - 5)\).
(a) Find the eigenvalues of \(A\) and the multiplicities of each.
(b) For each eigenvalue found above, give a basis for the corresponding eigenspace.

11. Let 
\[
V = \left\{ \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \in \mathbb{R}^3 : \mu_1 = 0 \right\}
\]
and 
\[
W = \left\{ \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \in \mathbb{R}^3 : \mu_2 = 0 \right\}
\]
(a) Prove (using the definition of subspace) that \(V\) is a subspace of \(\mathbb{R}^3\) and that \(W\) is a subspace of \(\mathbb{R}^3\).
(b) Show that \(V \cup W\) is not a subspace of \(\mathbb{R}^3\).

12. (a) Let \(A\) and \(B\) be \(n \times n\) matrices such that \(B\) is invertible. Prove that \(\det(B^{-1}AB) = \det A\).
(b) An \(n \times n\) matrix \(Q\) is called orthogonal if \(Q^TQ = I_n\). Prove that if \(Q\) is orthogonal, then \(\det Q = \pm 1\).

13. Explain why 
\[
\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -11 \\ 4 \end{bmatrix} \right\}
\]
is not a basis for \(\mathbb{R}^3\).

14. Let \(A\) be the matrix 
\[
\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}
\]
Show that $\det A = (b-a)(c-a)(c-b)$.

15. Determine the dimensions of (a) $\text{Col } A$ (b) $\text{Null } A$ (c) $\text{Row } A$ and (d) $\text{Null } A^T$, if

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ -1 & -2 & 2 & -2 \\ 2 & 3 & 0 & 3 \end{bmatrix}.$$ 

16. Show that each set is not a subspace of the appropriate $R^n$.

(a) \[
\left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in R^2 : u_1 u_2 = 0 \right\}.
\]

(b) \[
\left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in R^2 : u_1^2 + u_2^2 \leq 1 \right\}.
\]

17. Classify each statement as true or false and give a brief justification of your answer.

(a) There are some subspaces of $R^n$ that do not contain $0$.

(b) A vector $v$ is in $\text{Col } A$ if and only if $Av = 0$.

(c) A vector $v$ is in $\text{Row } A$ if and only if $A^T x = v$ is consistent.

(d) Every subspace of $R^n$ has a basis.

18. Classify each statement as true or false and give a brief justification of your answer.

(a) If $Ax = 0$ has a unique solution then the nullspace of $A$ is non-empty.

(b) If $u_1$, $u_2$ and $u_3$ belongs to a subspace $W$ of $R^n$ then $5u_1 + 3u_2 - u_3$ also belongs to $W$.

(c) A square matrix is invertible if and only of $\det A \neq 0$.

(d) If $A$ is a $5 \times 2$ matrix, then the nullspace of $A$ is never $\{0\}$.

19. Show that

$$B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$
is a basis for the subspace

\[
V = \left\{ \begin{bmatrix} 4t \\ s + t \\ -3s + t \end{bmatrix} \in \mathbb{R}^3 : s \text{ and } t \text{ are scalars} \right\}.
\]

20. Prove that if \( \lambda \) is an eigenvalue of the matrix \( A \), then \( \lambda^5 \) is an eigenvalue of the matrix \( A^5 \).

21. Let

\[
A = \begin{bmatrix} 1 & 1 & -1 & -2 \\ -1 & -2 & 1 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix}
\]

(i) Find a basis for the column space of \( A \)

(ii) Find a basis for the null space of \( A \).

22. Compute the determinant of

\[
\begin{bmatrix} 1 & 1 & x \\ -1 & 7 & y \\ 2 & 8 & z \end{bmatrix}
\]

by cofactor expansion along the third column.

23. Use Cramer’s rule (no credit for other methods!) to solve the following system of linear equations.

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= 2 \\
x_1 + x_3 &= 3 \\
x_1 + x_2 - x_3 &= 1
\end{align*}
\]

24. Let

\[
A = \begin{bmatrix} 5 & 4 \\ 7 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}
\]

Compute:

(a) \( \det(ABA^3B^8) \)

(b) \( \det(ABA^{-3}A^4B^{-10}) \)
25. Find a basis for the following subspace

\[ \text{Span} \left\{ \begin{bmatrix} 4 \\ 6 \\ -10 \end{bmatrix}, \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}, \begin{bmatrix} 21 \\ 6 \\ 0 \end{bmatrix} \right\} \]

26. Explain why

\[ \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\} \]

is not a generating set for \( \mathbb{R}^3 \).

27. Find the eigenvalues of the following matrix, and determine a basis for each eigenspace.

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & -2 & -3 & 3 \\ -6 & 0 & 1 & -3 \\ -6 & 0 & 0 & -2 \end{bmatrix} \]

28. Show that

\[ \begin{bmatrix} 4 & -5 \\ 3 & -2 \end{bmatrix} \]

has no real eigenvalues.

29. Classify each statement as true or false and give a brief justification of your answer.

(a) If \( A \) is an \( m \times n \) matrix then \( \dim \text{Null} A + \dim \text{Col} A = m \)

(b) The rank of a matrix \( A \) is equal to the rank of \( A^T \).

(c) If \( A \) is a 10 \( \times \) 10 matrix and \( \text{rank} \ A = 7 \) then 0 is a root of the characteristic polynomial of \( A \).

(d) If \( \mathbf{v} \) is an eigenvector of a matrix, then there is a unique eigenvalue of the matrix corresponding to \( \mathbf{v} \).

30. Let \( A \) be a certain 3 \( \times \) 3 matrix, and you know that \( A^4 = O \), where \( O \) is the zero-matrix. Show that the only eigenvalue of \( A \) is 0.

31. Classify each statement as true or false and give a brief justification of your answer.

(a) If \( \mathbf{v} \) is an eigenvector of matrix \( A \) then \( c \mathbf{v} \) is also an eigenvector for any scalar \( c \).
(b) A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if the equation $(A - \lambda I_n)x = 0$ has a non-zero solution.

(c) The rank of any matrix equals the dimension of its row space.

(d) If $v$ is an eigenvector of matrix $A$ then $cv$ is also an eigenvector for any non-zero scalar $c$.

32. Determine whether the matrix is invertible, and if it is, find its inverse:

$$
\begin{bmatrix}
1 & 0 & 1 \\
2 & -1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
$$