1. Determine whether the matrix is invertible. If so find, its inverse.

\[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 5 & 5 \\
1 & 3 & 1
\end{bmatrix}
\]

Solutions of 1: First we try to bring the matrix into reduced row-echelon form (of course, if it is not invertible, we can stop after phase I, declaring that it is not invertible, otherwise we continue to the end).

First apply the elementary row operations \( r_2 - 2r_1 \rightarrow r_2 \) and \( r_3 - r_1 \rightarrow r_3 \) to make the (2,1) and the (3,1) entries 0, getting:

\[
\begin{bmatrix}
1 & 3 & 2 \\
0 & -1 & 1 \\
0 & 0 & -1
\end{bmatrix}
\]

This is already in row-echelon form, so we know that the matrix is invertible and we must go on to the reduced-row-echelon form. Now we go bottom-up.

First me have to make all pivots 1, so we do \(-r_2 \rightarrow r_2\), \(-r_3 \rightarrow r_3\) getting:

\[
\begin{bmatrix}
1 & 3 & 2 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\]

Doing \( r_2 + r_3 \rightarrow r_2 \) and \( r_1 - 2r_3 \rightarrow r_1 \) yields

\[
\begin{bmatrix}
1 & 3 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and doing \( r_1 - 3r_2 \rightarrow r_1 \) gives:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

This is the reduced row-echelon-form and since it is the identity matrix \( I_3 \) the matrix is indeed invertible (of course we already knew that!, from phase I).

Now let’s collect, in order all the elementary row operations that we performed:

(i) \( r_2 - 2r_1 \rightarrow r_2 \)

(ii) \( r_3 - r_1 \rightarrow r_3 \)

(iii) \(-r_2 \rightarrow r_2\)
(iv) $-r_3 \rightarrow r_3$
(v) $r_2 + r_3 \rightarrow r_2$
(vi) $r_1 - 2r_3 \rightarrow r_1$
(vii) $r_1 - 3r_2 \rightarrow r_1$

Applying (i) and (ii) to the identity matrix gives
\[
\begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\]

Applying (iii) and (iv) gives
\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & -1 & 0 \\
1 & 0 & -1
\end{bmatrix}
\]

Applying (v) and (vi) yields
\[
\begin{bmatrix}
-1 & 0 & 2 \\
3 & -1 & -1 \\
1 & 0 & -1
\end{bmatrix}
\]

Applying (vii) yields:
\[
\begin{bmatrix}
-10 & 3 & 5 \\
3 & -1 & -1 \\
1 & 0 & -1
\end{bmatrix}
\]

**Ans. to 1:** The matrix is invertible and its inverse is the matrix
\[
\begin{bmatrix}
-10 & 3 & 5 \\
3 & -1 & -1 \\
1 & 0 & -1
\end{bmatrix}
\]

**Comments:** Only about 40% of the people got it completely, but most people did it the right way.

2. Use the algorithm for computing $A^{-1}B$, where
\[A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\]

**Sol. of 2:** We bring the matrix $A$ to reduced-row-echelon form.

First we do $r_2 + 2r_1 \rightarrow r_2$ to get
\[
\begin{bmatrix}
-1 & 2 \\
0 & 1
\end{bmatrix}
\]
This is already in row-echelon-form, so $A$ is indeed invertible. Continuing to phase II we first do $-r_1 \rightarrow r_1$, getting: 

$$
\begin{bmatrix}
1 & -2 \\
0 & 1
\end{bmatrix}
$$

followed by $r_1 + 2r_2 \rightarrow r_1$ yielding 

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
$$

So, as expected, the reduced-row-echelon form of $A$ is the identity matrix $I_2$.

Let’s record the elementary-row-operations that got us here, in order

(i) $r_2 + 2r_1 \rightarrow r_2$

(ii) $-r_1 \rightarrow r_1$

(iii) $r_1 + 2r_2 \rightarrow r_1$

Now apply these operations, in that order to the matrix $B$.

Applying operation (i) (namely $r_2 + 2r_1 \rightarrow r_2$) gives

$$
\begin{bmatrix}
4 & -1 \\
9 & 0
\end{bmatrix}
$$

Applying operation (ii) (namely $-r_1 \rightarrow r_1$) yields

$$
\begin{bmatrix}
-4 & 1 \\
9 & 0
\end{bmatrix},
$$

and finally, applying operation (iii) (namely $r_1 + 2r_2 \rightarrow r_1$) yields

$$
\begin{bmatrix}
14 & 1 \\
9 & 0
\end{bmatrix}.
$$

Ans. to 2:

$$
A^{-1}B = \begin{bmatrix}
14 & 1 \\
9 & 0
\end{bmatrix}.
$$

Comments: About 60% of the people got the correct answer. Another 20% used the right method, but got the wrong answer. Quite a few people did it the long way. They first found $A^{-1}$ like in problem 1, and then they did the matrix multiplication $A^{-1}B$. This is a waste of time! Of course for a 2 $\times$ 2 matrix it is not that difficult, but try to do it for a 5 $\times$ 5 matrix. Multiplying two five-by-five matrices takes forever, so please use the Gaussian elimination method, remember it is the same effort as finding the inverse, $A^{-1}$ but instead of starting with the identity matrix $I_n$, you start with the matrix $B$. 