Solutions to the Attendance Quiz for Nov. 4, 2010

1. A matrix and a vector are given. Show that the vector is an eigenvector of the matrix, and determine the corresponding eigenvalue.

\[
A = \begin{bmatrix}
-9 & -8 & 5 \\
7 & 6 & -5 \\
-6 & -6 & 4
\end{bmatrix}, \quad \begin{bmatrix}
3 \\
-2 \\
1
\end{bmatrix}.
\]

**Sol. to 1:** We multiply the matrix \(A\) by the given vector and see whether we get a multiple of that vector.

\[
A = \begin{bmatrix}
-9 & -8 & 5 \\
7 & 6 & -5 \\
-6 & -6 & 4
\end{bmatrix}
\begin{bmatrix}
3 \\
-2 \\
1
\end{bmatrix} = \begin{bmatrix}
(-9)(3) + (-8)(-2) + (5)(1) \\
(7)(3) + (6)(-2) + (-5)(1) \\
(-6)(3) + (-6)(-2) + (4)(1)
\end{bmatrix}
= \begin{bmatrix}
-27 + 16 + 5 \\
21 - 12 - 5 \\
-18 + 12 + 4
\end{bmatrix}
= \begin{bmatrix}
-6 \\
4 \\
-2
\end{bmatrix}.
\]

Obviously this vector is a multiple (by \(-2\)) to the original vector \(\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\):

\[
\begin{bmatrix}
-6 \\
4 \\
-2
\end{bmatrix} = (-2) \begin{bmatrix}
3 \\
-2 \\ 1
\end{bmatrix},
\]

so the proposed vector is indeed an eigenvector of the matrix \(A\) and the corresponding eigenvalue is \(-2\).

2. Below a matrix and a scalar \(\lambda\) are given. Show that \(\lambda\) is an eigenvector of the matrix and determine a basis for its eigenspace.

\[
A = \begin{bmatrix}
-11 & 14 \\
-7 & 10
\end{bmatrix}, \quad \lambda = -4
\]

**Sol. of 2:** We have to see whether the equation \(Ax = (-4)x\) has a non-zero solution, or in other words, we have to solve the system \((A - (-4)I_2)x = 0\). If the only solution is the zero vector, then it is not an eigenvalue. On the other-hand if we can find a non-zero solution then it is, and the set of solutions will be the eigenspace.

\[
A - (-4)I_2 = \begin{bmatrix}
-11 & 14 \\
-7 & 10
\end{bmatrix} + 4 \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
-11 + 4 & 14 \\
-7 & 10 + 4
\end{bmatrix} = \begin{bmatrix}
-7 & 14 \\
-7 & 14
\end{bmatrix}
\]

Doing Gaussian elimination we get

\[
\begin{bmatrix}
-7 & 14 \\
-7 & 14
\end{bmatrix} \rightarrow \begin{bmatrix}
r_2 - r_1 \rightarrow r_2 \\
r_1 \rightarrow (-1/7)r_1 \rightarrow r_1
\end{bmatrix} = \begin{bmatrix}
1 & -2 \\
0 & 0
\end{bmatrix}.
\]
This is in **reduced-row-echelon form**. In everyday notation this is:

\[ x_1 - 2x_2 = 0 \]

So the general solution is:

\[ x_1 = 2x_2 \]
\[ x_2 = x_2 \quad (\text{free}) \]

In vector notation this is

\[
\begin{bmatrix}
  x_1 \\
  x_2 
\end{bmatrix} =
\begin{bmatrix}
  2x_2 \\
  x_2 
\end{bmatrix} = x_2 \begin{bmatrix}
  2 \\
  1 
\end{bmatrix}.
\]

So the vector \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) is an eigenvector, but so are all its multiples, and the eigenspace is

\[
\text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.
\]

**Ans. to 2:** \( \lambda = -4 \) is indeed an eigenvalue and a basis for its eigenspace is

\[
\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.
\]

**Note:** Don’t confuse the eigenspace that is a subspace that has infinitely many inhabitants, with its basis that only has finitely many members, in this example, just one, since the dimension of the eigenspace is 1.