Solutions to the Attendance Quiz for Nov. 18, 2010

Version of Nov. 29., 2010 (Correcting the final answer, thanks to Charlie Cawczynski who won \$2)

1. (a) Diagonalize the matrix

$$A = \begin{bmatrix} -2 & -2 \\ 6 & 5 \end{bmatrix} \quad .$$

Sol. of 1a): We first find the **eigenvalues**, by finding the **characteristic polynomial**:

$$p(t) = \det(A - tI_2) = \det \begin{bmatrix} -2-t & -2 \\ 6 & 5-t \end{bmatrix} = (-2-t)(5-t)-(-2)(6) = (t+2)(t-5)+12 = t^2-3t-10+12 = t^2-3t+2$$

So the equation $p(t) = 0$, for this matrix, is:

$$t^2 - 3t + 2 = 0 \quad .$$

Factoring, we get:

$$(t-1)(t-2) = 0 \quad .$$

This gives the roots $t = 1$ and $t = 2$. These are the eigenvalues. So we have done half of the problem, the D part:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad .$$

In order to find the P part of the answer, we have to find eigenvectors corresponding to each of these two eigenvalues.

For $t = 1$ we have to solve the system

$$\begin{bmatrix} -2-1 & -2 \\ 6 & 5-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

Which is

$$\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

In everyday notation this is:

$$-3x_1 - 2x_2 = 0$$

$$6x_1 + 4x_2 = 0$$

These equations are multiples of each other, so we only need the first one. Choosing x_1 as the **free** variable (you are welcome to pick x_2) we see that $x_2 = (-3/2)x_1$. So the general solution is

$$x_1 = x_1 \quad (free)$$

$$x_2 = -\frac{3}{2}x_1 \quad .$$

In vector notation, this is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{3}{2}x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \quad .$$

So a basis for the eigenspace is the one-element set:

$$\left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$$

But a multiple of an eigenvector by any (non-zero!) number is still an eigenvector, so multiplying by 3 we get the eigenvector

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad .$$

This is the **first** column of P.

For $t = 2$ we have to solve the system

$$\begin{bmatrix} -2-2 & -2 \\ 6 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

Which is

$$\begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

In everyday notation this is:

$$-4x_1 - 2x_2 = 0$$

$$6x_1 + 3x_2 = 0$$

These equations are multiples of each other, so we only need the first one. Choosing x_1 as the **free** variable (you are welcome to pick x_2) we see that $x_2 = -2x_1$. So the general solution is

$$x_1 = x_1 \quad (free)$$

$$x_2 = -2x_1 \quad .$$

In vector notation, this is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad .$$

So a basis for the eigenspace is the one-element set:

$$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

This is the **second** column of P.

Ans. to 1a):

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad , \quad P = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \quad .$$

(b) Use (a) to compute A^{10} (Hint $2^{10} = 1024$).

For this problem we also need P^{-1}. For a 2×2 matrix it is easiest to use the following formula for the inverse of a matrix:

$$If \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad then \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad .$$

Since $\det(P) = -1$ we have

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \quad .$$

(Note that, by coincidence P^{-1} is the same as P, this is a fluke! Usually it does not happen).

In order to compute A^{10} we do $PD^{10}P^{-1}$.

D^{10} is easy:

$$D^{10} = \begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \quad .$$

So

$$A^{10} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3072 & -2048 \end{bmatrix} = \begin{bmatrix} -3068 & -2046 \\ 6138 & 4093 \end{bmatrix} \quad .$$

Ans. to 1b):

$$A^{10} = \begin{bmatrix} -3068 & -2046 \\ 6138 & 4093 \end{bmatrix} \quad .$$