Version of Nov. 29., 2010 (Correcting the final answer, thanks to Charlie Cawczynski who won $2)

1. (a) Diagonalize the matrix

\[ A = \begin{bmatrix} -2 & -2 \\ 6 & 5 \end{bmatrix} \]

**Sol. of 1a):** We first find the eigenvalues, by finding the characteristic polynomial:

\[ p(t) = \det(A-tI_2) = \det \begin{bmatrix} -2-t & -2 \\ 6 & 5-t \end{bmatrix} = (-2-t)(5-t)-(-2)(6) = (t+2)(t-5)+12 = t^2-3t-10+12 = t^2-3t+2 \]

So the equation \( p(t) = 0 \), for this matrix, is:

\[ t^2 - 3t + 2 = 0 \]

**Factoring**, we get:

\[ (t-1)(t-2) = 0 \]

This gives the roots \( t = 1 \) and \( t = 2 \). These are the eigenvalues. So we have done half of the problem, the \( D \) part:

\[ D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \]

In order to find the \( P \) part of the answer, we have to find eigenvectors corresponding to each of these two eigenvalues.

For \( t = 1 \) we have to solve the system

\[ \begin{bmatrix} -2-1 & -2 \\ 6 & 5-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Which is

\[ \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

In everyday notation this is:

\[-3x_1 - 2x_2 = 0 \]
\[6x_1 + 4x_2 = 0 \]

These equations are multiples of each other, so we only need the first one. Choosing \( x_1 \) as the **free** variable (you are welcome to pick \( x_2 \)) we see that \( x_2 = (-3/2)x_1 \). So the general solution is

\[ x_1 = x_1 \quad (\text{free}) \]
\[ x_2 = -\frac{3}{2}x_1 \].
In vector notation, this is:
\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{3}{2}x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}.
\]

So a basis for the eigenspace is the one-element set:
\[
\left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}
\]

But a multiple of an eigenvector by any (non-zero!) number is still an eigenvector, so multiplying by 3 we get the eigenvector
\[
\begin{bmatrix} 2 \\ -3 \end{bmatrix}.
\]

This is the first column of $P$.

For $t = 2$ we have to solve the system
\[
\begin{bmatrix} -2 & -2 \\ 6 & 5 - 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Which is
\[
\begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

In everyday notation this is:
\[
-4x_1 - 2x_2 = 0
\]
\[
6x_1 + 3x_2 = 0
\]

These equations are multiples of each other, so we only need the first one. Choosing $x_1$ as the free variable (you are welcome to pick $x_2$) we see that $x_2 = -2x_1$. So the general solution is
\[
x_1 = x_1 \quad (\text{free})
\]
\[
x_2 = -2x_1.
\]

In vector notation, this is:
\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}.
\]

So a basis for the eigenspace is the one-element set:
\[
\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}
\]

This is the second column of $P$.

Ans. to 1a):
\[
D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}.
\]
(b) Use (a) to compute $A^{10}$ (Hint $2^{10} = 1024$).

For this problem we also need $P^{-1}$. For a $2 \times 2$ matrix it is easiest to use the following formula for the inverse of a matrix:

$$
If \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad then \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
$$

Since $\det(P) = -1$ we have

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}.$$

(Note that, by coincidence $P^{-1}$ is the same as $P$, this is a fluke! Usually it does not happen).

In order to compute $A^{10}$ we do $PD^{10}P^{-1}$.

$D^{10}$ is easy:

$$D^{10} = \begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}.$$

So

$$A^{10} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3072 & -2048 \end{bmatrix} = \begin{bmatrix} -3068 & -2046 \\ 6138 & 4093 \end{bmatrix}.$$

Ans. to 1b):

$$A^{10} = \begin{bmatrix} -3068 & -2046 \\ 6138 & 4093 \end{bmatrix}.$$