1. Let $v$ be a non-zero vector in $R^2$, and let $A = vv^T$ ($A$ is a $2 \times 2$ matrix.)

(a) Show that $v$ is an eigenvector of $A$. What is the eigenvalue?

**Sol. to 1a):** First way: The concrete way: Write

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Then

$$A = vv^T = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1^2 & v_1v_2 \\ v_2v_1 & v_2^2 \end{bmatrix}$$

Now do $Av$ and see what happens:

$$Av = \begin{bmatrix} v_1^2 & v_1v_2 \\ v_2v_1 & v_2^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1^3 + v_1v_2^2 \\ v_2^3 + v_2v_2^2 \end{bmatrix}.$$  

Factoring both components we get that this is

$$\begin{bmatrix} (v_1^2 + v_2^2)v_1 \\ (v_1^2 + v_2^2)v_2 \end{bmatrix}.$$  

This means that we can take out $v_1^2 + v_2^2$ and get

$$Av = (v_1^2 + v_2^2) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (v_1^2 + v_2^2)v.$$  

Hooray!: we proved that $Av$ is a scalar-multiple of $v$ and the magnification factor, alias eigenvalue, is $v_1^2 + v_2^2$.

**Second way:** (the abstract way valid for every $R^n$):

$$Av = (vv^T)v = v(v^Tv),$$

by associativity. But unlike $vv^T$ that is an $n \times n$ matrix, $v^Tv$ is a $1 \times 1$ matrix, alias scalar. So we can move it to the left and get:

$$Av = (v^Tv)v.$$  

This means that $Av = \lambda v$ with $\lambda = v^Tv$, and that’s the eigenvalue.

(b) What is the rank of $A$? What is the other eigenvalue of $A$?

**Sol. of 1b):** The rank is 1 (not 2, too many people said 2!). Since both rows (and for $R^n$ all $n$ rows) are multiples of the same row, the dimension of the row space is 1. By the famous
theorem, the dimension of the column space, is the same as the dimension of the row space, so it is also 1. But the dimension of the column space is the same as the rank of the matrix. Since the rank is 1 the matrix is not invertible, and this means that 0 is an eigenvalue. So the other eigenvalue is 0.

2. Prove that if $\lambda$ is an eigenvalue of the matrix $A$, then $\lambda^5$ is an eigenvalue of the matrix $A^5$.

Sol. of 2: By the definition of eigenvalue, there is a non-zero vector $v$ (called the eigenvector) such that

$$Av = \lambda v .$$

Multiplying from the left by $A$ we get

$$AAv = A(\lambda v) = \lambda (Av) ,$$

and this is legal, since $\lambda$ is a scalar, and you can move it around. Of course $AA = A^2$, and we use $Av = \lambda v$ once again, we have:

$$A^2v = \lambda (Av) = \lambda^2 v ,$$

So $\lambda^2$ is an eigenvalue of $A^2$ with the same eigenvector, namely $v$. Now keep doing the same thing (multiply both sides by $A$) getting that

$\lambda^3$ is an eigenvector of $A^3$ (you do it!)

And again! getting that

$\lambda^4$ is an eigenvector of $A^4$ (you do it!)

And again! getting that

$\lambda^5$ is an eigenvector of $A^5$ (you do it!)

still with the very same eigenvector, $v$. 