Solutions to the Attendance Quiz for Nov. 11, 2010

1. Let $\mathbf{v}$ be a non-zero vector in R^2, and let $A = \mathbf{v}\mathbf{v}^T$ (A is a a 2×2 natrix.)

(a) Show that $\mathbf{v}$ is an eigenvector of A. What is the eigenvalue?

Sol. to 1a): First way: The concrete way: Write

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Then

$$A = \mathbf{v}\mathbf{v}^T = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1^2 & v_1 v_2 \\ v_2 v_1 & v_2^2 \end{bmatrix}$$

Now do $A\mathbf{v}$ and see what happens:

$$A\mathbf{v} = \begin{bmatrix} v_1^2 & v_1 v_2 \\ v_2 v_1 & v_2^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1^3 + v_1 v_2^2 \\ v_2^2 v_2 + v_2^3 \end{bmatrix} \quad .$$

Factoring both components we get that this is

$$\begin{bmatrix} (v_1^2 + v_2^2) v_1 \\ (v_1^2 + v_2^2) v_2 \end{bmatrix} \quad .$$

This means that we can **take out** $v_1^2 + v_2^2$ and get

$$A\mathbf{v} = (v_1^2 + v_2^2) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (v_1^2 + v_2^2)\mathbf{v} \quad .$$

Hooray!: we proved that $A\mathbf{v}$ is a scalar-multiple of $\mathbf{v}$ and the magnification factor, alias **eigenvalue**, is $v_1^2 + v_2^2$.

Second way: (the abstract way valid for every R^n):

$$A\mathbf{v} = (\mathbf{v}\mathbf{v}^T)\mathbf{v} = \mathbf{v}(\mathbf{v}^T\mathbf{v}) \quad ,$$

by associativity. But unlike $\mathbf{v}\mathbf{v}^T$ that is an $n \times n$ matrix, $\mathbf{v}^T\mathbf{v}$ is a 1×1 matrix, **alias scalar**. So we can move it to the left and get:

$$A\mathbf{v} = (\mathbf{v}^T\mathbf{v})\mathbf{v} \quad ,$$

This means that $A\mathbf{v} = \lambda\mathbf{v}$ with $\lambda = \mathbf{v}^T\mathbf{v}$, and that's the eigenvalue.

(b) What is the rank of A? What is the other eigenvalue of A?

Sol. of 1b): The rank is 1 (not 2, too many people said 2!). Since both rows (and for R^n all n rows) are multiples of the same row, the **dimension** of the **row space** is 1. By the famous

theorem, the dimension of the **column space**, is the same as the dimension of the row space, so it is also 1. **But** the dimension of the column space is the same as the **rank** of the matrix. Since the rank is 1 the matrix is not invertible, and this means that 0 is an eigenvalue. So the other eigenvalue is 0.

2. Prove that if λ is an eigenvalue of the matrix A, then λ^5 is an eigenvalue of the matrix A^5.

Sol. of 2: By the definition of eigenvalue, there is a **non-zero** vector $\mathbf{v}$ (called the **eigenvector**) such that

$$A\mathbf{v} = \lambda\mathbf{v} \quad .$$

Multiplying from the left by A we get

$$AA\mathbf{v} = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) \quad ,$$

and this is legal, since λ is a scalar, and you can move it around. Of course $AA = A^2$, and we use $A\mathbf{v} = \lambda\mathbf{v}$ once again, we have:

$$A^2\mathbf{v} = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v} \quad ,$$

So λ^2 is an eigenvalue of A^2 with the **same eigenvector**, namely $\mathbf{v}$. Now keep doing the same thing (multiply both sides by A) getting that

λ^3 is an eigenvector of A^3 (you do it!)

And again! getting that

λ^4 is an eigenvector of A^4 (you do it!)

And again! getting that

λ^5 is an eigenvector of A^5 (you do it!)

still with the very same **eigenvector**, $\mathbf{v}$.