Solutions to the Attendance Quiz for Dec. 2, 2010

1. (a) Apply the Gram-Schmidt process to replace the given linearly independent set $S$ by an orthogonal set of non-zero vectors with the same span.

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

**Sol. of 1:** The data is

$$u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Gram-Schmidt says

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{||v_1||^2} v_1.$$  

So

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$  

We have

$$u_2 \cdot v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = (1)(1) + (-2)(-1) + (1)(0) = 1 + 2 + 0 = 3,$$

and

$$||v_1||^2 = (1)^2 + (-2)^2 + (1)^2 = 6.$$  

Putting it together, we have:

$$v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -\frac{1}{2} \end{bmatrix}.$$  

**Ans. to 1a:** An orthogonal basis for the span of $S$ is:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \right\}.$$  

**Comments:** About 60% of the people did it perfectly. Another 20% did it the right way, but messed up the calculations.
(b) Obtain an orthonormal set with the same span as \( S \).

**Sol. of 1b):** We divide these two vectors by their norms. We have
\[
||v_1|| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6},
\]
\[
||v_2|| = \sqrt{\left(\frac{1}{2}\right)^2 + (0)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}.
\]
So
\[
w_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{2\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix},
\]
\[
w_2 = \frac{v_2}{||v_2||} = \sqrt{2} \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}.
\]

**Ans. to 1b:** An orthonormal basis for the span of \( S \) is:
\[
\begin{bmatrix}
\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\
-\frac{2\sqrt{6}}{6} & 0 \\
\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2}
\end{bmatrix}.
\]

**Comments:** About \( \%50 \) of the people did it perfectly. Another \( \%20 \) did it the right way, but messed up the calculations.

**Note:** I didn’t have a chance to teach the topic of the second problem, below, but it is very important. I will cover it at the beginning of next class. Nevertheless, please read the question and the solution carefully, so you will be prepared to absorb it quickly when I teach it on Monday.

2. (a) Let \( A \) be the matrix whose columns are the vectors in \( S \) in the above problem. Use the answer to that problem to determine the matrices \( Q \) and \( R \) in a QR factorization of \( A \).

**Sol. of 2:** The \( Q \) part is immediate, it is the matrix whose two columns are \( w_1 \) and \( w_2 \). So
\[
Q = \begin{bmatrix}
\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\
-\frac{2\sqrt{6}}{6} & 0 \\
\frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{2}
\end{bmatrix}.
\]
To get the \( R \) part, we write
\[
a_1 = r_{11}w_1
\]
\[
a_2 = r_{12}w_1 + r_{22}w_2
\].
So

\[ r_{11} = a_1 \cdot w_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{-\sqrt{6}}{\sqrt{6}} \end{bmatrix} = \sqrt{6} \]

\[ r_{12} = a_2 \cdot w_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{-\sqrt{6}}{\sqrt{6}} \end{bmatrix} = (1)\left(\frac{\sqrt{6}}{6}\right) + (-1)(\frac{-2\sqrt{6}}{6}) + (0)(\frac{\sqrt{6}}{6}) = \frac{\sqrt{6}}{2} \]

\[ r_{22} = a_2 \cdot w_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = (1)(\frac{\sqrt{2}}{2}) + (-1)(0) + (0)(\frac{-\sqrt{2}}{2}) = \frac{\sqrt{2}}{2} \]

So we have

\[ R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{6} & \frac{\sqrt{6}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \]

Ans. to 2a):

\[ Q = \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\ \frac{-2\sqrt{6}}{\sqrt{6}} & 0 \\ \frac{\sqrt{6}}{\sqrt{6}} & \frac{-\sqrt{2}}{2} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{6} & \frac{\sqrt{6}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \]

(b) Verify that indeed \( A = QR \).

Sol. to 2b): You do it!