Solution to Attendance Quiz for Dec. 13, 2010

1. Obtain the solution of least norm to the equation

$$x_1 - 3x_2 + 2x_3 = 5 \quad .$$

There are **three** different ways of doing this, each of them perfectly correct. I will do it my favorite way, via calculus.

The first step is to use Gaussian elimination (in this problem there is nothing to do) to get the **general solution**:

$$x_1 = 5 + 3x_2 - 2x_3$$

$$x_2 = x_2 \, (free)$$

$$x_3 = x_3 \, (free)$$

And in **vector form**:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 + 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

Instead of minimizing the **norm** it is easier to minimize the **square of the norm**. The norm-square of the general solution is

$$x_1^2 + x_2^2 + x_3^2 = (5 + 3x_2 - 2x_3)^2 + x_2^2 + x_3^2 \quad .$$

Let's give it a name, $R(x_2, x_3)$. We have to minimize R. Recall that this is done by taking the partial derivatives R_{x_2} and R_{x_3}, setting them both equal to zero and solving for x_2, x_3 (so we still need linear algebra, but only of the high-school kind, two equations and two unknowns). We have

$$R = (5 + 3x_2 - 2x_3)^2 + x_2^2 + x_3^2 \quad ,$$

So

$$R_{x_2} = 2(5+3x_2-2x_3)(3)+2x_2 = 6(5+3x_2-2x_3)+2x_2 = 30+18x_2-12x_3+2x_2 = 30+20x_2-12x_3 \quad ,$$

$$R_{x_3} = 2(5+3x_2-2x_3)(-2)+2x_3 = (-4)(5+3x_2-2x_3)+2x_3 = -20-12x_2+8x_3+2x_3 = -20-12x_2+10x_3 \quad .$$

So we have to solve the system

$$30 + 20x_2 - 12x_3 = 0 \quad ,$$

$$-20 - 12x_2 + 10x_3 = 0$$

Rearranging (and dividing by 2):

$$10x_2 - 6x_3 = -15 \quad ,$$

$$-6x_2 + 5x_3 = 10$$

The solution is $x_2 = -\frac{15}{14}$, $x_3 = \frac{5}{7}$. (you do it!). Now all we have to do is plug-in these values in the above general solution getting:

$$x_1 = 5 + 3x_2 - 2x_3 = 5 + 3(-\frac{15}{14}) - 2(\frac{5}{7}) = \frac{5}{14}$$

$$x_2 = -\frac{15}{14}$$

$$x_3 = \frac{5}{7}$$

Ans. to 1:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} \\ -\frac{15}{14} \\ \frac{5}{7} \end{bmatrix} \quad .$$

Comments: Another way of doing this problem is to write the general solution in the usual vector form

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Define W as the span of the two vectors in front of the free variables:

$$W = Span \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad ,$$

then form the matrix C whose columns are these vectors. Then use the famous formula for P_W ($P_W = C(C^T C)^{-1} C^T$), and then find $P_W \mathbf{b}$ where $\mathbf{b}$ is the numerical vector in the general solution: $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$, and finally do $\mathbf{b} - P_W \mathbf{b}$. Yet another way is to find $P_W(\mathbf{b})$ via Gram-Schmidt etc.