Solutions to MATH 250 (1), Dr. Z. , Exam 2, Mon., Nov. 15, 2010, 8:40-10:00am, SEC 202

1. (a) (6 points) Find an LU decomposition of the matrix

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ -1 & 2 & 0 & -2 & -2 \\ 2 & -1 & 7 & -1 & 1 \end{bmatrix}$$

Sol. of 1a: We try to bring to row-echelon from, only performing operations of the form $r_i + cr_j \to r_i$.

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ -1 & 2 & 0 & -2 & -2 \\ 2 & -1 & 7 & -1 & 1 \end{bmatrix} \begin{matrix} r_2 + r_1 \to r_2 \\ \longrightarrow \\ r_3 - 2r_1 \to r_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & -3 & -5 \end{bmatrix} \begin{matrix} r_3 - r_2 \to r_3 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix}$$

This is U, to get L we look at the operations:

$$r_2 + r_1 \to r_2 \quad \textit{implies} \quad L_{2,1} = -1$$

$$r_3 - 2r_1 \to r_3 \quad \textit{implies} \quad L_{3,1} = 2$$

$$r_3 - r_2 \to r_3 \quad \textit{implies} \quad L_{3,2} = 1$$

This means that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad .$$

Ans. to 1a):

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad . \quad , \quad \begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -2 & -4 \end{bmatrix}$$

(b) (4 points) Use the results of part (a) to solve the system

$$x_1 - x_2 + 2x_3 + x_4 + 3x_5 = -4$$

$$-x_1 + 2x_2 \quad - 2x_4 - 2x_5 = 9$$

$$2x_1 - x_2 + 7x_3 - x_4 + x_5 = -2$$

Sol. of 1b: Out system is $LU\mathbf{x} = \mathbf{b}$ where

$$\mathbf{b} = \begin{bmatrix} -4 \\ 9 \\ -2 \end{bmatrix} \quad .$$

We write $U\mathbf{x} = \mathbf{y}$, and first solve $L\mathbf{y} = \mathbf{b}$. In other words we have to solve

$$y_1 = -4 \quad ,$$

$$-y_1 + y_2 = 9 \quad ,$$

$$2y_1 + y_2 + y_3 = -2 \quad .$$

Going from **top to bottom** we get, of course $y_1 = -4$, then $y_2 = 9 + y_1 = 5$ and finally $y_3 = -2 - 2y_1 - y_2 = -2 - 2(-4) - 5 = 1$. So

$$\mathbf{y} = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} \quad .$$

Now we solve $U\mathbf{x} = \mathbf{y}$, getting the system

$$x_1 - x_2 + 2x_3 + x_4 + 3x_5 = -4 \quad ,$$

$$x_2 + 2x_3 - x_4 + x_5 = 5 \quad ,$$

$$x_3 - 3x_4 - 6x_5 = 1 \quad .$$

Going **bottom-up**, we have:

$$x_3 = 1 + 2x_4 + 6x_5$$

From the second equation we get:

$$x_2 = 5-2x_3+x_4-x_5 = 5-2(1+2x_4+6x_5)+x_4-x_5 = 5-2-4x_4-12x_5+x_4-x_5 = 3-3x_4-13x_5 \quad ,$$

From the first equation we get

$$x_1 = -4 + x_2 - 2x_3 - x_4 - 3x_5 = -4 + (3 - 3x_4 - 13x_5) - 2(1 + 2x_4 + 6x_5) - x_4 - 3x_5$$

$$= -4 + 3 - 3x_4 - 13x_5 - 2 - 4x_4 - 12x_5 - x_4 - 3x_5 = -3 - 8x_4 - 28x_5 \quad .$$

Combining, we get that the general solution of the system is

$$x_1 = -3 - 8x_4 - 28x_5 \quad ,$$

$$x_2 = 3 - 3x_4 - 13x_5 \quad ,$$

$$x_3 = 1 + 2x_4 + 6x_5 \quad ,$$

$$x_4 = x_4 \quad (free),$$

$$x_5 = x_5 \quad (free).$$

2. (10 points) Compute the determinant by using elementary row operations (no credit for other methods)

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix}$$

Sol. to 2:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix} \begin{matrix} r_2 - 2r_1 \to r_2 \\ r_3 + 4r_1 \to r_3 \\ \longrightarrow \\ r_4 - 3r_1 \to r_4 \end{matrix} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 4 & -4 \end{bmatrix} \begin{matrix} r_3 - r_2 \to r_3 \\ \longrightarrow \\ r_4 - r_2 \to r_4 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 6 & -2 \end{bmatrix} r_4 - 2r_3 \to r_4 \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Now this matrix is **upper-triangular**, so its determinant is the product of the diagonal entries: (1)(1)(3)(6). Since we **only** used row operation of the kind $r_i + cr_j \to r_i$, the determinant of the original matrix is exactly the same.
Ans. of 2: 18.

3. (10 points) For what values of c is the given matrix **not** invertible.

$$\begin{bmatrix} -1 & 1 & 1 \\ 3 & -2 & -c \\ 0 & c & -10 \end{bmatrix}$$

Sol. of 2: We first compute the determinant. The easiest is via **first-column** cofactor expansion:

$$\det \begin{bmatrix} -1 & 1 & 1 \\ 3 & -2 & -c \\ 0 & c & -10 \end{bmatrix} = (-1) \cdot \det \begin{bmatrix} -2 & -c \\ c & -10 \end{bmatrix} - (3) \cdot \det \begin{bmatrix} 1 & 1 \\ c & -10 \end{bmatrix} + 0 \det \cdot \begin{bmatrix} 1 & 1 \\ = 2 & -c \end{bmatrix}$$

$$= (-1)((-2)(-10) - (-c)(c)) - 3((1)(-10) - (1)(c)) =$$

$$(-1)(20 + c^2) - 3(-10 - c) = -20 - c^2 + 30 + 3c = -c^2 + 3c + 10 = -(c^2 - 3c - 10)$$

To find out when the matrix is **not** invertible, you set it equal to 0:

$$-(c^2 - 3c - 10) = 0 \quad ,$$

and solve for c. Factoring gives

$$-(c - 5)(c + 2) = 0 \quad ,$$

whose solutions are $c = 5$ and $c = -2$.

Ans. to 3: The matrix is not invertible when $c = 5$ and the matrix is not invertible when $c = -2$.

4. Explain why the following set is a subspace of R^4 and find a basis for it.

$$\left\{ \begin{bmatrix} r+s+2t \\ r-s \\ 3r+2s+5t \\ -2r+3s+t \end{bmatrix} \in R^4 : r, s, \ and \ t \ are \ scalars \right\}$$

Sol. of 4: By separating the r, s and t contributions, we can write this set as

$$\left\{ r \begin{bmatrix} 1 \\ 1 \\ 3 \\ -2 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \end{bmatrix} : r, s, \ and \ t \ are \ scalars \right\} = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\} .$$

Being a span of a finite set of vectors, this is automatically a **subspace**, and the set of these three vectors forms a **generating set**. To find a **basis**, we use **Gaussian elimination** on the matrix whose columns are these three vectors.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \\ -2 & 3 & 1 \end{bmatrix} .$$

Doing $r_2 - r_1 \to r_1, r_3 - 3r_1 \to r_3, r_4 + 2r_1 \to r_4$ we get:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & -1 & -1 \\ 0 & 5 & 5 \end{bmatrix} .$$

Doing: $-\frac{1}{2}r_2 \to r_2$, $-r_3 \to r_3$, and $-\frac{1}{5}r_4 \to r_4$ gives:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} .$$

Doing $r_3 - r_2 \to r_3, r_4 - r_2 \to r_4$ gives

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

This is in row-echelon form (no need to go all the way). The first two columns happen to be **pivot columns**, so by the **column-correspondence property** the first two columns of the **original** matrix, form a basis.

Ans. to 4): This set of vectors is a subspace since it is the span of the above three vectors, and a basis for that subspace is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \right\} \quad .$$

Note: In this problem, you can also note **by inspection** that the third vector is a sum of the first two vectors, so it can be kicked-out. Since the remaining two vectors are **not** multiples of each other, they are linearly independent, and hence form a basis.

5. Explain why the following sets in R^3 are **not** subspaces
(a) (5 points)

$$\left\{ \begin{bmatrix} r \\ 2r \\ 3r \end{bmatrix} \in R^3 : r \geq 0 \right\}$$

Sol. of 5a): All we need is to find an example where one of the subspace properties is violated. Taking $r = 1$ shows that the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ belongs to our set. But multiplying it by any negative number, for example by -1 will give the vector $(-1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$, that is definitely **not** in our set, since all components of all vectors in the set have **non-negative** components, so there is no way that it can belong.

(b)

$$\left\{ \begin{bmatrix} 1+r \\ 2+r \\ 3+r \end{bmatrix} \in R^3 : r \, is \ a \ scalar \right\}$$

Sol. of 5b): This is even easier. The zero-vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does **not** belong to our set, since if it did we would have to come up with a number r such that

$$\begin{bmatrix} 1+r \\ 2+r \\ 3+r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Spelling it out we need an r such that

$$1 + r = 0 \quad ,$$

$$2 + r = 0 \quad ,$$

$$3 + r = 0 \quad ,$$

and clearly there is no such r, becuase according to the first equation, $r = -1$, according to the second, $r = -2$ and according to the third, $r = -3$, and this is clearly **impossible**. So already the first property of a subspace that $\mathbf{0}$ belongs to it is violated.

6. Let

$$V = \left\{ \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \in R^2 : \nu_1 = 0 \right\}$$

$$W = \left\{ \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \in R^2 : \nu_2 = 0 \right\}$$

(a) (5 points) Prove (using the definition of subspace) that V is a subspace of R^2 and that W is a subspace of R^2.

Sol. of 5a): To prove that V is a subspace,
In plain English, V is the set of all vectors in R^2 whose first component is 0. we check the three properties of the subspace.

(i) $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ belongs to V because its first component is 0.
(ii) A constant multiple of any vector whose first component is zero still has that property.
(iii) The sum of two vectors whose first component is 0 still has that property since $0+0 = 0$.

The proof for W is similar with "first" replaced by "second".

(b) Show that $V \cup W$ is **not** a subspace of R^2.

Sol. of 5b): Proof by contradiction. Suppuse that $V \cup W$ is a subspace.
Since $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$ it is also true that $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V \cup W$
Since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$ it is also true that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V \cup W$
If $V \cup W$ would have been a subspace, than the sum of these vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ would belong to $V \cup W$, but this is not true! This vector belongs to neither V (since its first component is not 0) nor to W (since its second component is not 0), so it can't belong to $V \cup W$. QED.

7. (10 points) Explain why

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -11 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ -11 \\ 13 \\ 22 \end{bmatrix} \right\}$$

is not a basis for R^4.

Sol. of 7: A basis of R^4 must have exactly 4 elements, and this set has five, so there is no way that it could be basis.

8. (10 points, 2.5 points each) Determine the dimensions of (a) $Col\,A$ (b) $Null\,A$ (c) $Row\,A$ and (d) $Null\,A^T$, if

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & 1 & -2 \\ -1 & 0 & 4 \end{bmatrix}$$

Sol. of 8: We first find the **rank** by doing row-operations. Doing $r_2 \leftrightarrow r_1$ we get:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 3 & 1 & -2 \\ -1 & 0 & 4 \end{bmatrix}$$

Doing $r_3 - 3r_1 \to r_3$, $r_4 + r_1 \to r_4$ gives:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 0 & -5 & 7 \\ 0 & 2 & 1 \end{bmatrix}$$

Doing $r_3 - 5r_2 \to r_3$ and $r_4 + 2r_2 \to r_4$ gives

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} .$$

This is in **row-echelon form** and we see that each column has a pivot, so the **rank** is 3. Since this is a 4×3 matrix, $m = 4, n = 3$. Now we can answer all four questions
(a) Dimension of $Col\,A$ is the rank, so it is 3.
(b) Dimension of $Null\,A$ is $n - rank = 3 - 3 = 0$.
(c) Dimension of $Row\,A$ is the same as the dimension of $Col\,A$, alias the rank, so it is 3.
(d) The dimension of $Null\,A^T$ is $m - rank = 4 - 3 = 1$.

9. (10 ponts. 2.5 each) Classify each statement as true or false and give a brief justification of your answer.

(a) If $A\mathbf{x} = \mathbf{0}$ has a unique solution then the nullspace of A is empty.

Sol. of 9a): **False**. The corrected statement is
"If $A\mathbf{x} = \mathbf{0}$ has a unique solution then the nullspace of A is the zero-subspace."
which is "almost empty" but does contain one element, the zero-vector $\mathbf{0}$.

(b) If $\mathbf{u}$ and $\mathbf{v}$ belongs to a subspace W of R^n then $5\mathbf{u} + 11\mathbf{v}$ also belongs to W.

Sol. of 9b): **True**. By the property of subspace, since $5\mathbf{u} + 11\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) A square matrix is invertible if and only of $\det A = 0$.

Sol. of 9c): **False**. The corrected statement is:
"A square matrix is invertible if and only of $\det A \neq 0$.

(d) If A is a 10×13 matrix, then the nullspace of A is not $\{\mathbf{0}\}$.

Sol. of 9d): **True**. The nullity is at least 3 so the dimension of the nullspace is at least 3.

10. (10 points) Prove that if λ is an eigenvalue of the matrix A, then λ^2 is an eigenvalue of the matrix A^2.

Sol. of 10: By definition of eigenvalue, there exists a vector, calld **eigenvector**, let's denote it by $\mathbf{v}$, such that

$$A\mathbf{v} = \lambda\mathbf{v} \quad .$$

Left-multipling this equation by A we get a new equation:

$$AA\mathbf{v} = A(\lambda\mathbf{v}) \quad .$$

Since $AA = A^2$ and matrices commute with scalars ($A\lambda = \lambda A$) we have

$$A^2\mathbf{v} = \lambda(A\mathbf{v}) \quad .$$

Using the top equation $A\mathbf{v} = \lambda\mathbf{v}$ one more time (on the right side) we get

$$A^2\mathbf{v} = \lambda(\lambda\mathbf{v}) \quad .$$

which means:

$$A^2\mathbf{v} = \lambda^2\mathbf{v} \quad .$$

This means that λ^2 is an eigenvalue of A^2 (with the same eigenvector!). QED.