Solutions to MATH 250 (1), Dr. Z , Exam 2, Mon., Nov. 15, 2010, 8:40-10:00am, SEC 202

1. (a) (6 points) Find an LU decomposition of the matrix

\[
\begin{bmatrix}
1 & -1 & 2 & 1 & 3 \\
-1 & 2 & 0 & -2 & -2 \\
2 & -1 & 7 & -1 & 1
\end{bmatrix}
\]

Sol. of 1a: We try to bring to row-echelon form, only performing operations of the form \( r_i + cr_j \rightarrow r_i \).

\[
\begin{bmatrix}
1 & -1 & 2 & 1 & 3 \\
-1 & 2 & 0 & -2 & -2 \\
2 & -1 & 7 & -1 & 1
\end{bmatrix}
\]

\[ r_2 + r_1 \rightarrow r_2 \]
\[ r_3 - 2r_1 \rightarrow r_3 \]

\[
\begin{bmatrix}
1 & -1 & 2 & 1 & 3 \\
0 & 1 & 2 & -1 & -1 \\
0 & 1 & 3 & -3 & -5
\end{bmatrix}
\]

This is \( U \), to get \( L \) we look at the operations:

\[ r_2 + r_1 \rightarrow r_2 \text{ implies } L_{2,1} = -1 \]
\[ r_3 - 2r_1 \rightarrow r_3 \text{ implies } L_{3,1} = 2 \]
\[ r_3 - r_2 \rightarrow r_3 \text{ implies } L_{3,2} = 1 \]

This means that

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
2 & 1 & 1
\end{bmatrix}
\]

Ans. to 1a):

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
2 & 1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & -1 & 2 & 1 & 3 \\
0 & 1 & 2 & -1 & -1 \\
0 & 0 & 1 & -2 & -4
\end{bmatrix}
\]

(b) (4 points) Use the results of part (a) to solve the system

\[
x_1 - x_2 + 2x_3 + x_4 + 3x_5 = -4 \\
-x_1 + 2x_2 - 2x_4 - 2x_5 = 9 \\
2x_1 - x_2 + 7x_3 - x_4 + x_5 = -2
\]

Sol. of 1b: Out system is \( LUx = b \) where

\[
b = \begin{bmatrix}
-4 \\
9 \\
-2
\end{bmatrix}
\]
We write $U\mathbf{x} = \mathbf{y}$, and first solve $L\mathbf{y} = \mathbf{b}$. In other words we have to solve

$$y_1 = -4 \quad ,$$

$$-y_1 + y_2 = 9 \quad ,$$

$$2y_1 + y_2 + y_3 = -2 \quad .$$

Going from top to bottom we get, of course $y_1 = -4$, then $y_2 = 9 + y_1 = 5$ and finally $y_3 = -2 - 2y_1 - y_2 = -2 - 2(-4) - 5 = 1$. So

$$\mathbf{y} = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} \quad .$$

Now we solve $U\mathbf{x} = \mathbf{y}$, getting the system

$$x_1 - x_2 + 2x_3 + x_4 + 3x_5 = -4 \quad ,$$

$$x_2 + 2x_3 - x_4 + x_5 = 5 \quad ,$$

$$x_3 - 3x_4 - 6x_5 = 1 \quad .$$

Going bottom-up, we have:

$$x_3 = 1 + 2x_4 + 6x_5$$

From the second equation we get:

$$x_2 = 5 - 2x_3 + x_4 - x_5 = 5 - 2(1 + 2x_4 + 6x_5) + x_4 - x_5 = 5 - 2 - 4x_4 - 12x_5 + x_4 - x_5 = 3 - 3x_4 - 13x_5 \quad ,$$

From the first equation we get

$$x_1 = -4 + x_2 - 2x_3 - x_4 - 3x_5 = -4 + (3 - 3x_4 - 13x_5) - 2(1 + 2x_4 + 6x_5) - x_4 - 3x_5$$

$$= -4 + 3 - 3x_4 - 13x_5 - 2 - 4x_4 - 12x_5 - x_4 - 3x_5 = -3 - 8x_4 - 28x_5 \quad .$$

Combining, we get that the general solution of the system is

$$x_1 = -3 - 8x_4 - 28x_5 \quad ,$$

$$x_2 = 3 - 3x_4 - 13x_5 \quad ,$$

$$x_3 = 1 + 2x_4 + 6x_5 \quad ,$$

$$x_4 = x_4 \quad (\text{free}),$$

$$x_5 = x_5 \quad (\text{free}).$$
2. (10 points) Compute the determinant by using elementary row operations (no credit for other methods)

\[
\begin{bmatrix}
1 & -1 & 2 & 1 \\
2 & -1 & -1 & 4 \\
-4 & 5 & -10 & -6 \\
3 & -2 & 10 & -1
\end{bmatrix}
\]

Sol. to 2:

\[
\begin{bmatrix}
1 & -1 & 2 & 1 \\
2 & -1 & -1 & 4 \\
-4 & 5 & -10 & -6 \\
3 & -2 & 10 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 2 & 1 \\
0 & 1 & -5 & 2 \\
0 & 0 & 3 & -4 \\
0 & 0 & 0 & 6
\end{bmatrix}
\]

Now this matrix is upper-triangular, so its determinant is the product of the diagonal entries: \((1)(1)(3)(6)\). Since we only used row operation of the kind \(r_i + cr_j \rightarrow r_i\), the determinant of the original matrix is exactly the same.

Ans. of 2: 18.
3. (10 points) For what values of $c$ is the given matrix not invertible.

\[
\begin{bmatrix}
-1 & 1 & 1 \\
3 & -2 & -c \\
0 & c & -10
\end{bmatrix}
\]

**Sol. of 2:** We first compute the determinant. The easiest is via *first-column* cofactor expansion:

\[
\det \begin{bmatrix}
-1 & 1 & 1 \\
3 & -2 & -c \\
0 & c & -10
\end{bmatrix}
= (-1) \cdot \det \begin{bmatrix}
-2 & -c \\
c & -10
\end{bmatrix} - (3) \cdot \det \begin{bmatrix}
1 & 1 \\
c & -10
\end{bmatrix} + 0 \cdot \det \begin{bmatrix}
1 & 1 \\
0 & c
\end{bmatrix}
\]

\[
= (-1)((-2)(-10) - (-c)(c)) - 3((1)(-10) - (1)(c))
\]

\[
= (-1)(20 + c^2) - 3(-10 - c) = -20 - c^2 + 30 + 3c = -c^2 + 3c + 10 = -(c^2 - 3c - 10)
\]

To find out when the matrix is not invertible, you set it equal to 0:

\[-(c^2 - 3c - 10) = 0 ,
\]

and solve for $c$. Factoring gives

\[-(c - 5)(c + 2) = 0 ,
\]

whose solutions are $c = 5$ and $c = -2$.

**Ans. to 3:** The matrix is not invertible when $c = 5$ and the matrix is not invertible when $c = -2.$
4. Explain why the following set is a subspace of $\mathbb{R}^4$ and find a basis for it.

\[
\begin{bmatrix}
  r + s + 2t \\
r - s \\
3r + 2s + 5t \\
-2r + 3s + t
\end{bmatrix} \in \mathbb{R}^4: r, s, \text{ and } t \text{ are scalars}
\]

Sol. of 4: By separating the $r$, $s$, and $t$ contributions, we can write this set as

\[
\begin{bmatrix}
  r \\
  \frac{1}{3} \\
-2
\end{bmatrix} + s
\begin{bmatrix}
  1 \\
-\frac{1}{2} \\
  2
\end{bmatrix} + t
\begin{bmatrix}
  2 \\
  0 \\
  5
\end{bmatrix} : r, s, \text{ and } t \text{ are scalars}
\]

\[
= \text{Span}\left\{ \begin{bmatrix}
  1 \\
-\frac{1}{3} \\
-2
\end{bmatrix}, \begin{bmatrix}
  1 \\
-1 \\
 3
\end{bmatrix}, \begin{bmatrix}
  0 \\
 2 \\
 5
\end{bmatrix} \right\}
\]

Being a span of a finite set of vectors, this is automatically a subspace, and the set of these three vectors forms a generating set. To find a basis, we use Gaussian elimination on the matrix whose columns are these three vectors.

\[
\begin{bmatrix}
  1 & 1 & 2 \\
  1 & -1 & 0 \\
  3 & 2 & 5 \\
-2 & 3 & 1
\end{bmatrix}
\]

Doing $r_2 - r_1 \rightarrow r_1, r_3 - 3r_1 \rightarrow r_3, r_4 + 2r_1 \rightarrow r_4$ we get:

\[
\begin{bmatrix}
  1 & 1 & 2 \\
  0 & -2 & -2 \\
  0 & -1 & -1 \\
  0 & 5 & 5
\end{bmatrix}
\]

Doing: $-\frac{1}{2}r_2 \rightarrow r_2, -r_3 \rightarrow r_3$, and $-\frac{1}{5}r_4 \rightarrow r_4$ gives:

\[
\begin{bmatrix}
  1 & 1 & 2 \\
  0 & 1 & 1 \\
  0 & 1 & 1 \\
  0 & 1 & 1
\end{bmatrix}
\]

Doing $r_3 - r_2 \rightarrow r_3, r_4 - r_2 \rightarrow r_4$ gives

\[
\begin{bmatrix}
  1 & 1 & 2 \\
  0 & 1 & 1 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}
\]

This is in row-echelon form (no need to go all the way). The first two columns happen to be pivot columns, so by the column-correspondence property the first two columns of the original matrix, form a basis.
Ans. to 4): This set of vectors is a subspace since it is the span of the above three vectors, and a basis for that subspace is

\[
\begin{bmatrix}
1 \\
1 \\
3 \\
-2
\end{bmatrix},
\begin{bmatrix}
1 \\
-1 \\
2 \\
3
\end{bmatrix},
\begin{bmatrix}
1 \\
-1 \\
2 \\
3
\end{bmatrix}
\].

Note: In this problem, you can also note by inspection that the third vector is a sum of the first two vectors, so it can be kicked-out. Since the remaining two vectors are not multiples of each other, they are linearly independent, and hence form a basis.
5. Explain why the following sets in $\mathbb{R}^3$ are \textbf{not} subspaces

(a) (5 points)
\[
\begin{cases}
      r \\
      2r \\
      3r
\end{cases} \in \mathbb{R}^3 : r \geq 0
\]

\textbf{Sol. of 5a):} All we need is to find an example where one of the subspace properties is violated. Taking $r = 1$ shows that the vectors \[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\] belongs to our set. But multiplying it by any negative number, for example by $-1$ will give the vector \[
\begin{bmatrix}
-1 \\
-2 \\
-3
\end{bmatrix},
\]
that is definitely \textbf{not} in our set, since all components of all vectors in the set have \textbf{non-negative} components, so there is no way that it can belong.

(b) \[
\begin{cases}
      1 + r \\
      2 + r \\
      3 + r
\end{cases} \in \mathbb{R}^3 : \text{$r$ is a scalar}
\]

\textbf{Sol. of 5b):} This is even easier. The zero-vector $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does \textbf{not} belong to our set, since if it did we would have to come up with a number $r$ such that
\[
\begin{bmatrix}
1 + r \\
2 + r \\
3 + r
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
Spelling it out we need an $r$ such that
\[
1 + r = 0,
\]
\[
2 + r = 0,
\]
\[
3 + r = 0,
\]
and clearly there is no such $r$, because according to the first equation, $r = -1$, according to the second, $r = -2$ and according to the third, $r = -3$, and this is clearly \textbf{impossible}. So already the first property of a subspace that $0$ belongs to it is violated.
6. Let

\[ V = \left\{ \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \in \mathbb{R}^2 : \nu_1 = 0 \right\} \]

\[ W = \left\{ \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \in \mathbb{R}^2 : \nu_2 = 0 \right\} \]

(a) (5 points) Prove (using the definition of subspace) that \( V \) is a subspace of \( \mathbb{R}^2 \) and that \( W \) is a subspace of \( \mathbb{R}^2 \).

**Sol. of 5a):** To prove that \( V \) is a subspace, in plain English, \( V \) is the set of all vectors in \( \mathbb{R}^2 \) whose first component is 0. we check the three properties of the subspace.

(i) \( 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) belongs to \( V \) because its first component is 0.

(ii) A constant multiple of any vector whose first component is zero still has that property.

(iii) The sum of two vectors whose first component is 0 still has that property since 0+0 = 0.

The proof for \( W \) is similar with “first” replaced by “second”.

(b) Show that \( V \cup W \) is not a subspace of \( \mathbb{R}^2 \).

**Sol. of 5b):** Proof by contradiction. Suppose that \( V \cup W \) is a subspace.

Since \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V \) it is also true that \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V \cup W \)

Since \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W \) it is also true that \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V \cup W \)

If \( V \cup W \) would have been a subspace, than the sum of these vectors \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) would belong to \( V \cup W \), but this is not true! This vector belongs to neither \( V \) (since its first component is not 0) nor to \( W \) (since its second component is not 0), so it can’t belong to \( V \cup W \). QED.
7. (10 points) Explain why

\[
\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -11 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ -11 \\ 13 \\ 22 \end{bmatrix} \right\}
\]

is not a basis for \( \mathbb{R}^4 \).

**Sol. of 7:** A basis of \( \mathbb{R}^4 \) must have exactly 4 elements, and this set has five, so there is no way that it could be basis.
8. (10 points, 2.5 points each) Determine the dimensions of (a) \text{Col } A \text{ (b) } \text{Null } A \text{ (c) } \text{Row } A \text{ and (d) } \text{Null } A^T, \text{ if}

\[ A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & 1 & -2 \\ -1 & 0 & 4 \end{bmatrix} \]

\text{Sol. of 8:} \text{ We first find the rank by doing row-operations. Doing } r_2 \leftrightarrow r_1 \text{ we get:}

\[ \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 3 & 1 & -2 \\ -1 & 0 & 4 \end{bmatrix} \]

Doing \( r_3 - 3r_1 \rightarrow r_3, r_4 + r_1 \rightarrow r_4 \) gives:

\[ \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 0 & 5 & 7 \\ 0 & 2 & 1 \end{bmatrix} \]

Doing \( r_3 - 5r_2 \rightarrow r_3 \) and \( r_4 + 2r_2 \rightarrow r_4 \) gives

\[ \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}. \]

This is in row-echelon form and we see that each column has a pivot, so the rank is 3.

Since this is a 4 × 3 matrix, \( m = 4, n = 3 \). Now we can answer all four questions

(a) Dimension of \text{Col } A \text{ is the rank, so it is 3.}
(b) Dimension of \text{Null } A \text{ is } n - \text{rank} = 3 - 3 = 0.
(c) Dimension of \text{Row } A \text{ is the same as the dimension of } \text{Col } A, \text{ alias the rank, so it is 3.}
(d) The dimension of \text{Null } A^T \text{ is } m - \text{rank} = 4 - 3 = 1.
9. (10 ponts. 2.5 each) Classify each statement as true or false and give a brief justification of your answer.

(a) If $A\mathbf{x} = \mathbf{0}$ has a unique solution then the nullspace of $A$ is empty.

**Sol. of 9a): False.** The corrected statement is

“If $A\mathbf{x} = \mathbf{0}$ has a unique solution then the nullspace of $A$ is the zero-subspace.”

which is “almost empty” but does contain one element, the zero-vector $\mathbf{0}$.

(b) If $\mathbf{u}$ and $\mathbf{v}$ belongs to a subspace $W$ of $\mathbb{R}^n$ then $5\mathbf{u} + 11\mathbf{v}$ also belongs to $W$.

**Sol. of 9b): True.** By the property of subspace, since $5\mathbf{u} + 11\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) A square matrix is invertible if and only of $\det A = 0$.

**Sol. of 9c): False.** The corrected statement is:

“A square matrix is invertible if and only of $\det A \neq 0$.

(d) If $A$ is a $10 \times 13$ matrix, then the nullspace of $A$ is not $\{\mathbf{0}\}$.

**Sol. of 9d): True.** The nullity is at least 3 so the dimension of the nullspace is at least 3.
10. (10 points) Prove that if $\lambda$ is an eigenvalue of the matrix $A$, then $\lambda^2$ is an eigenvalue of the matrix $A^2$.

**Sol. of 10:** By definition of eigenvalue, there exists a vector, called **eigenvector**, let’s denote it by $v$, such that

$$Av = \lambda v$$

Left-multiplying this equation by $A$ we get a new equation:

$$AAv = A(\lambda v)$$

Since $AA = A^2$ and matrices commute with scalars ($A\lambda = \lambda A$) we have

$$A^2v = \lambda(Av)$$

Using the top equation $Av = \lambda v$ one more time (on the right side) we get

$$A^2v = \lambda(\lambda v)$$

which means:

$$A^2v = \lambda^2 v$$

This means that $\lambda^2$ is an eigenvalue of $A^2$ (with the same eigenvector!). QED.