SOLUTIONS to MATH 250 (2), Dr. Z. , Exam 1, Thurs., Oct. 14, 2010, 10:20-11:40am, ARC 204

1. (10 pts. altogether)
(a) (7 pts.) Determine if the given system is consistent, and if so, find its general solution.

$$2x_1 - 2x_2 + 4x_3 = 1 \quad ,$$
$$-4x_1 + 4x_2 - 8x_3 = -2 \quad .$$

Sol. of 1(a): The system in **matrix** notation is

$$\begin{bmatrix} 2 & -2 & 4 \\ -4 & 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad .$$

The **augmented matrix** is:

$$\begin{bmatrix} 2 & -2 & 4 & 1 & 1 \\ -4 & 4 & -8 & -2 & \end{bmatrix}$$

Doing **Gaussian elemination gives**:

$$\begin{bmatrix} 2 & -2 & 4 & 1 & 1 \\ -4 & 4 & -8 & -2 & \end{bmatrix} \begin{matrix} r_2 + 2r_1 \to r_2 \\ \to \end{matrix} \begin{bmatrix} 2 & -2 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & \end{bmatrix} \begin{matrix} (1/2)r_1 \to r_1 \\ \to \end{matrix} \begin{bmatrix} 1 & -1 & 2 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & \end{bmatrix} \quad .$$

This is in **reduced row echelon form**.

There is only one **pivot column**, the first, making x_1 a **basic variable**. Columns 2 and 3 are free columns, making x_2, x_3 **free variables**.

In everyday language this means:

$$x_1 - 2x_2 + 2x_3 = 1/2 \quad .$$

Solving for the basic variable x_1, we get $x_1 = 1/2 + x_2 - 2x_3$, and for the free variables (since they free to do what they want) $x_2 = x_2, x_3 = x_3$. So the **general solution** in high-school notation is:

$$x_1 = 1/2 + x_2 - 2x_3$$
$$x_2 = x_2$$
$$x_3 = x_3$$

(b) (3 points) Express the general solution of part a) in **vector notation**.

Sol. of 1(b): Using the general solution above we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 + x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad .$$

Ans. to 1b):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad .$$

2. (10 pts.) Let

$$\mathcal{S} = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad , \quad \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \},$$

determine whether the set $\mathcal{S}$ is linearly independent or linearly dependent. In case it is linearly dependent, write the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ explicitly as a non-trivial linear combination of the vectors in $\mathcal{S}$.

Sol. of 2): The easiest way to do this is **by inspection**. The third vector is the sum of the first two:

$$\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad .$$

This automatically means that $\mathcal{S}$ is **linearly dependent** (since you can express one of its members as a linear combination of other members). For getting the desired explicit relation, we move everthing to the right getting:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \quad .$$

Ans. to 2: $\mathcal{S}$ is linearly dependent and the explicit expression of $\mathbf{0}$ as a non-trivial linear combination of the vectors in $\mathcal{S}$ is:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \quad .$$

Note: In this problem we got lucky. In general we have to use Gaussian elimination to get it to **reduced row echelom form**. If you do it (you do it!) you would get

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad .$$

So the rank is 2 (less than 3), or equivalenty, we have an all-zeros-row, this means that the members of $\mathcal{S}$ are linearly dependent. Also from R we see that the third column is the sum of the first two. By the **column-correspondence property** the same is true for the original matrix whose columns were the given vectors in $\mathcal{S}$.

3. (10 pts altogether) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Calculate the following matrix products, if they are defined, or explain why they don't make sense.

(a) (4 points) BB^T

Sol. to 3a):

$$BB^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) (3 points) BA

Sol. to 3b): B is a 3×2 matrix and A is a 3×3 matrix. You can't do $(3 \times 2)(3 \times 3)$ since $2 \neq 3$.

Ans. to 3b): undefined.
(c) (3 points) C^3

Sol. to 3c):

$$C^2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$C^3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \quad .$$

Ans. to 3c):

$$C^3 = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$$

4. (10 pts.) For the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

compute the matrix A^{17}.

Sol. to 4): $A^2 = I_3$, so $A^{16} = I_3^8 = I_3$ and $A^{17} = I_3 A = A$.

Ans. to 4):

$$A^{17} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad .$$

5. (10 pts.) For the following matrix A finds its **reduced-row-echelon form**, R, and find an invertible matrix P such that $PA = R$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Sol. of 5: We first bring the matrix to **reduced row echelon form**, taking careful note of the elementary row operations:

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \underset{\rightarrow}{r_3 - r_1 \rightarrow r_3} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \underset{\rightarrow}{r_3 + r_2 \rightarrow r_3} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{\rightarrow}{r_1 - r_2 \rightarrow r_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad .$$

Now it is in reduced-row-echelon form. This is the first part of the **answer**, R. To get P we apply the above elementary row operations to the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\rightarrow}{r_3 - r_1 \rightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \underset{\rightarrow}{r_3 + r_2 \rightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \underset{\rightarrow}{r_1 - r_2 \rightarrow r_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad .$$

Ans. to 5:

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Note: Since A is not invertible (R is not I_3), there are more than one correct P. The R is always the same, regadless of the choice of the order of elementary row operations, but the P may be different. That's why it is good to check that $PA = R$, becuase there is more than one correct P that makes it come true.

6. (10 pts. altogether) In each case below, give an $m \times n$ matrix R in *reduced row echelon form* satisfying the given condition, or explain why it is impossible to do so.

(a)(4 pts) $m = 2$, $n = 3$ and the equation $R\mathbf{x} = \mathbf{c}$ has a solution for all $\mathbf{c}$.

(b) (4 pts) $m = 3$, $n = 2$ and the equation $R\mathbf{x} = \mathbf{c}$ has a unique solution for all $\mathbf{c}$.

(c) (2 pts) $m = 3$, $n = 3$ and the equation $R\mathbf{x} = \mathbf{0}$ has a unique solution.

Sol. to 6a): There are many "correct solutions". One of them is:

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Explanation: The system $R\mathbf{x} = \mathbf{c}$, in *high-school language* is:

$$x_1 + 2x_3 = c_1 \quad ,$$

$$x_2 + 3x_3 = c_2 \quad .$$

Obviously you can solve it for **any** choice of real numbers c_1, c_2. x_3 is a free variable, and the general solution is $x_1 = c_1 - 2x_3, x_2 = c_2 - 3x_3, x_3 = x_3$, so in this system there are **infinitely many** solutions for all $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, but this is besides the point. The question demanded there is **at least one** solution for any choice of $\mathbf{c}$ and of course this is true in this case.

(b) (4 pts) $m = 3$, $n = 2$ and the equation $R\mathbf{x} = \mathbf{c}$ has a unique solution for all $\mathbf{c}$.

Sol. to 6b): **Impossible**.

Explanation: Since the rank is ≤ 2 there must be (at least) one all-zeros row, and the matrix looks like

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The system $R\mathbf{x} = \mathbf{c}$ in everyday notation is

$$x_1 = c_1$$

$$x_2 = c_2$$

$$0 = c_3 \quad .$$

Whenever $c_3 \neq 0$ we get **nonsense** , i.e. the system is **inconsistent**, i.e. it has **no solutions**. So forget about "unique" solution. For many choices fo $\mathbf{c}$ (namely whenever $c_3 \neq 0$) it has **no** solutions!

(c) (2 pts) $m = 3$, $n = 3$ and the equation $R\mathbf{x} = \mathbf{0}$ has a unique solution.

Sol. to 6c):

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Explanation: The system $R\mathbf{x} = \mathbf{0}$, in *high-school language* is:

$$x_1 = 0 \quad ,$$

$$x_2 = 0 \quad ,$$

$$x_3 = 0 \quad .$$

This **system** is its **own solution**, and obviously has a unique solution $\mathbf{0}$.

7. (10 pts.) **Without first computing** A^{-1}, find $A^{-1}B$, if

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

Sol. of 7: We first use **Gaussian elimination** and bring A to **reduced-row-echelon-form**. If this would not be I_3 it would mean that the problem is a bad one, since then A^{-1} would not make sense (since then A will not be invertible). So if we trust the problem, then we already know that we should get $R = I_3$. **But** we need more than R, we need the sequence of elementary row operations that got us there.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \overset{r_3 - r_2 \to r_3}{\to} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{r_1 - r_2 \to r_1}{\to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad .$$

Hooray we got I_3. Now we **mimick** the same sequence of operations starting with B.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 \end{bmatrix} \overset{r_3 - r_2 \to r_3}{\to} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \end{bmatrix} \overset{r_1 - r_2 \to r_1}{\to} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \end{bmatrix} \quad .$$

Ans. to 7:

$$A^{-1}B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \end{bmatrix} \quad .$$

8. (10 pts. altogether , 2 each) **True** or **False**? Give a short explanation!
(a) Every system of linear equations has at least one solution.

Sol. to 8a): **False**. It may be inconsisent (have no solution). For example

$$x_1 = 1$$

$$x_1 = 2 \quad .$$

(b) If a matrix is in row-echelon form then the pivot entry of each pivot-column must be 1

Sol. to 8b): **False**. This is only a requirement for **reduced** row-echelon form.

(c) If A is an $m \times n$ matrix, then $A\mathbf{x} = \mathbf{b}$ is consistent for every b in R^m if and only if the rank of A is n.

Sol. to 8c): **False**. To make it true replace "rank of A is n" by "rank of A is m".

(d) If A and B are invertible $n \times n$ matrices then $A + B$ is invertible.

Sol. to 8d): **False**. For example take $A = I_2$ and $B = -I_2$.

(e) Every column of a matrix is a linear combination of its pivot columns.

Sol. to 8e): **True**. This follows from the **column-correspondence property**.

9. (10 pts.) Let $\mathbf{u_1}, \mathbf{u_2}, \ldots \mathbf{u_k}$ be vectors in R^n . Prove that the span of $\{\mathbf{u_1}, \mathbf{u_2}, \ldots \mathbf{u_k}\}$ is the same as the span of $\{\mathbf{u_1} + 2\mathbf{u_2}, \mathbf{u_2}, \ldots \mathbf{u_k}\}$.

Sol. of 9: The span of $\mathbf{u_1}, \mathbf{u_2}, \ldots \mathbf{u_k}$ consists of all **linear combinations** of $\mathbf{u_1}, \mathbf{u_2}, \ldots \mathbf{u_k}$, while the span of $\mathbf{u_1}+2\mathbf{u_2}, \mathbf{u_2}, \ldots \mathbf{u_k}$ consists of all **linear combinations** of $\mathbf{u_1}+2\mathbf{u_2}, \mathbf{u_2}, \ldots \mathbf{u_k}$.

We have to prove that every vector in the first set also belongs to the second set, and vice versa.

It is easier to prove the second statment first. A member of the second set has the following **format**:

$$c_1(\mathbf{u_1} + 2\mathbf{u_2}) + c_2\mathbf{u_2} + c_3\mathbf{u_3} + \ldots + c_k\mathbf{u_k} \quad ,$$

for *some* k real numbers $c_1, c_2, \ldots, c_k$. Using the algebra of vectors (opening-up parentheses and then collecting terms), this equals

$$c_1\mathbf{u_1} + 2c_1\mathbf{u_2} + c_2\mathbf{u_2} + c_3\mathbf{u_3} + \ldots + c_k\mathbf{u_k} = c_1\mathbf{u_1} + (2c_1 + c_2)\mathbf{u_2} + c_3\mathbf{u_3} + \ldots + c_k\mathbf{u_k} \quad .$$

But this is the right format for the first set, since whenever c_1, c_2 are real numbers, so is $2c_1 + c_2$, so this is a legitimate member of the first set.

In order to prove the first statement we use a **trick**. Write $\mathbf{u_1}$ as

$$\mathbf{u_1} = (\mathbf{u_1} + 2\mathbf{u_2}) - 2\mathbf{u_2} \quad .$$

Any linear combination

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + \ldots + c_k\mathbf{u_k}$$

can be written as

$$c_1[(\mathbf{u_1} + 2\mathbf{u_2}) - 2\mathbf{u_2}] + c_2\mathbf{u_2} + \ldots + c_k\mathbf{u_k} = c_1(\mathbf{u_1} + 2\mathbf{u_2}) + (c_2 - 2c_1)\mathbf{u_2} + \ldots + c_k\mathbf{u_k} \quad ,$$

and the latter is a member of the second set.

Since every vector of the second set also belongs to the first set, and vice versa, this means that these two sets are the same. QED.

Note: Very few people got it comletely. Quite a few people were on the right track. The above proof is only one correct way.

10. Compute the product of the partitioned matrix using block multiplication.

$$\begin{bmatrix} 1 & -1 \\ --- & --- \\ 3 & 1 \\ -1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & |2 & 3 & 0 \\ -1 & |2 & -1 & 2 \end{bmatrix}$$

Sol. of 10: We view the matrices as matrices of **blocks** and give them names:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} [\, B_1 \quad B_2 \,]$$

Where

$$A_1 = [\, 1 \quad -1 \,]$$

$$A_2 = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 1 & 2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 2 & 3 & 0 \\ 2 & -1 & 2 \end{bmatrix} \quad .$$

We first do the **matrix-product** (of a 2×1 matrix times a 1×2 matrix, getting a 2×2 matrix (*symbolically*):

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} [\, B_1 \quad B_2 \,] = \begin{bmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{bmatrix} \quad .$$

Now we have to do **four** matrix-products:

$$A_1 B_1 = [\, 1 \quad -1 \,] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [(1)(1) + (-1)(-1)] = [2] \quad .$$

$$A_1 B_2 = [\, 1 \quad -1 \,] \begin{bmatrix} 2 & 3 & 0 \\ 2 & -1 & 2 \end{bmatrix} = [\, 0 \quad 4 \quad -2 \,]$$

$$A_2 B_1 = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix}$$

$$A_2 B_2 = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 2 \\ 8 & -8 & 10 \\ 6 & 1 & 4 \end{bmatrix} =$$

Now we place everything on top getting

$$\begin{bmatrix} 2 & 0 & 4 & -2 \\ 2 & 8 & 8 & 2 \\ -6 & 8 & -8 & 10 \\ -1 & 6 & 1 & 4 \end{bmatrix} \quad .$$

This is the **answer**.