1. (10 pts. altogether)
(a) (7 pts.) Determine if the given system is consistent, and if so, find its general solution.

\[ \begin{align*}
2x_1 - 2x_2 + 4x_3 &= 1, \\
-4x_1 + 4x_2 - 8x_3 &= -2.
\end{align*} \]

Sol. of 1(a): The system in matrix notation is

\[
\begin{bmatrix}
2 & -2 & 4 \\
-4 & 4 & -8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-2
\end{bmatrix}.
\]

The augmented matrix is:

\[
\begin{bmatrix}
2 & -2 & 4 & 1 & 1 \\
-4 & 4 & -8 & -2
\end{bmatrix}
\]

Doing Gaussian elimination gives:

\[
\begin{bmatrix}
2 & -2 & 4 & 1 & 1 \\
-4 & 4 & -8 & -2
\end{bmatrix}
\xrightarrow{r_2 + 2r_1 \to r_2}
\begin{bmatrix}
2 & -2 & 4 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\xrightarrow{(1/2)r_1 \to r_1}
\begin{bmatrix}
1 & -1 & 2 & 1 & 1/2
\end{bmatrix}.
\]

This is in reduced row echelon form.

There is only one pivot column, the first, making \(x_1\) a basic variable. Columns 2 and 3 are free columns, making \(x_2, x_3\) free variables.

In everyday language this means:

\[ x_1 - 2x_2 + 2x_3 = 1/2. \]

Solving for the basic variable \(x_1\), we get \(x_1 = 1/2 + x_2 - 2x_3\), and for the free variables (since they free to do what they want) \(x_2 = x_2, x_3 = x_3\). So the general solution in high-school notation is:

\[ x_1 = 1/2 + x_2 - 2x_3 \\
x_2 = x_2 \\
x_3 = x_3 \]

(b) (3 points) Express the general solution of part a) in vector notation.

Sol. of 1(b): Using the general solution above we have

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1/2 + x_2 - 2x_3 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1/2 \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
x_2 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
-2 \\
0 \\
1
\end{bmatrix}.
\]

Ans. to 1b):
2. (10 pts.) Let 
\[ S = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \}, \]
determine whether the set \( S \) is linearly independent or linearly dependent. In case it is linearly dependent, write the zero vector \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) explicitly as a non-trivial linear combination of the vectors in \( S \).

**Sol. of 2):** The easiest way to do this is by inspection. The third vector is the sum of the first two:
\[ \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}. \]
This automatically means that \( S \) is linearly dependent (since you can express one of its members as a linear combination of other members). For getting the desired explicit relation, we move everything to the right getting:
\[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}. \]

**Ans. to 2:** \( S \) is linearly dependent and the explicit expression of \( 0 \) as a non-trivial linear combination of the vectors in \( S \) is:
\[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}. \]

**Note:** In this problem we got lucky. In general we have to use Gaussian elimination to get it to reduced row echelon form. If you do it (you do it!) you would get
\[ R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \]
So the rank is 2 (less than 3), or equivalently, we have an all-zeros-row, this means that the members of \( S \) are linearly dependent. Also from \( R \) we see that the third column is the sum of the first two. By the column-correspondence property the same is true for the original matrix whose columns were the given vectors in \( S \).
3. (10 pts altogether) Let

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \]

Calculate the following matrix products, if they are defined, or explain why they don’t make sense.

(a) (4 points) \(BB^T\)

\[ BB^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

(b) (3 points) \(BA\)

\[ B \text{ is a } 3 \times 2 \text{ matrix and } A \text{ is a } 3 \times 3 \text{ matrix. You can’t do } (3 \times 2)(3 \times 3) \text{ since } 2 \neq 3. \]

\[ \text{Ans. to 3b): undefined.} \]

(c) (3 points) \(C^3\)

\[ C^2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \]

\[ C^3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \]

\[ \text{Ans. to 3c):} \]

\[ C^3 = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \]
4. (10 pts.) For the matrix

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

compute the matrix \( A^{17} \).

**Sol. to 4):** \( A^2 = I_3 \), so \( A^{16} = I_3^8 = I_3 \) and \( A^{17} = I_3A = A \).

**Ans. to 4):**

\[
A^{17} = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
5. (10 pts.) For the following matrix $A$ finds its reduced-row-echelon form, $R$, and find an invertible matrix $P$ such that $PA = R$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

**Sol. of 5:** We first bring the matrix to reduced row echelon form, taking careful note of the elementary row operations:

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now it is in reduced-row-echelon form. This is the first part of the **answer**, $R$. To get $P$ we apply the above elementary row operations to the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

**Ans. to 5:**

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

**Note:** Since $A$ is not invertible ($R$ is not $I_3$), there are more than one correct $P$. The $R$ is always the same, regardless of the choice of the order of elementary row operations, but the $P$ may be different. That’s why it is good to check that $PA = R$, because there is more than one correct $P$ that makes it come true.
6. (10 pts. altogether) In each case below, give an \( m \times n \) matrix \( R \) in reduced row echelon form satisfying the given condition, or explain why it is impossible to do so.

(a) (4 pts) \( m = 2, n = 3 \) and the equation \( Rx = c \) has a solution for all \( c \).

(b) (4 pts) \( m = 3, n = 2 \) and the equation \( Rx = c \) has a unique solution for all \( c \).

(c) (2 pts) \( m = 3, n = 3 \) and the equation \( Rx = 0 \) has a unique solution.

**Sol. to 6a)**: There are many “correct solutions”. One of them is:

\[
R = \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 3 \\
\end{bmatrix}
\]

**Explanation**: The system \( Rx = c \), in high-school language is:

\[
x_1 + 2x_3 = c_1 ,
\]

\[
x_2 + 3x_3 = c_2 .
\]

Obviously you can solve it for any choice of real numbers \( c_1, c_2 \). \( x_3 \) is a free variable, and the general solution is \( x_1 = c_1 - 2x_3, x_2 = c_2 - 3x_3, x_3 = x_3 \), so in this system there are infinitely many solutions for all \( c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \), but this is besides the point. The question demanded there is at least one solution for any choice of \( c \) and of course this is true in this case.

(b) (4 pts) \( m = 3, n = 2 \) and the equation \( Rx = c \) has a unique solution for all \( c \).

**Sol. to 6b)**: Impossible.

**Explanation**: Since the rank is \( \leq 2 \) there must be (at least) one all-zeros row, and the matrix looks like

\[
R = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

The system \( Rx = c \) in everyday notation is

\[
x_1 = c_1
\]

\[
x_2 = c_2
\]

\[
0 = c_3 .
\]
Whenever $c_3 \neq 0$ we get nonsense, i.e. the system is inconsistent, i.e. it has no solutions. So forget about “unique” solution. For many choices for $c$ (namely whenever $c_3 \neq 0$) it has no solutions!

(c) (2 pts) $m = 3, n = 3$ and the equation $Rx = 0$ has a unique solution.

**Sol. to 6c):**

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Explanation:** The system $Rx = 0$, in high-school language is:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0.$$ 

This system is its own solution, and obviously has a unique solution $\mathbf{0}$. 
7. (10 pts.) Without first computing $A^{-1}$, find $A^{-1}B$, if

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

Sol. of 7: We first use Gaussian elimination and bring $A$ to reduced-row-echelon-form. If this would not be $I_3$ it would mean that the problem is a bad one, since then $A^{-1}$ would not make sense (since then $A$ will not be invertible). So if we trust the problem, then we already know that we should get $R = I_3$. But we need more than $R$, we need the sequence of elementary row operations that got us there.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Hooray we got $I_3$. Now we mimick the same sequence of operations starting with $B$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \end{bmatrix}.$$ 

Ans. to 7:

$$A^{-1}B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \end{bmatrix}.$$
8. (10 pts. altogether, 2 each) **True** or **False**? Give a short explanation!
   (a) Every system of linear equations has at least one solution.

   **Sol. to 8a:** **False.** It may be inconsistent (have no solution). For example
   \[
   x_1 = 1 \\
   x_1 = 2 
   \]

   (b) If a matrix is in row-echelon form then the pivot entry of each pivot-column must be 1

   **Sol. to 8b:** **False.** This is only a requirement for **reduced** row-echelon form.

   (c) If \( A \) is an \( m \times n \) matrix, then \( Ax = b \) is consistent for every \( b \) in \( \mathbb{R}^m \) if and only if the rank of \( A \) is \( n \).

   **Sol. to 8c:** **False.** To make it true replace “rank of \( A \) is \( n \)” by “rank of \( A \) is \( m \)”.

   (d) If \( A \) and \( B \) are invertible \( n \times n \) matrices then \( A + B \) is invertible.

   **Sol. to 8d:** **False.** For example take \( A = I_2 \) and \( B = -I_2 \).

   (e) Every column of a matrix is a linear combination of its pivot columns.

   **Sol. to 8e:** **True.** This follows from the **column-correspondence property**.
9. (10 pts.) Let \( u_1, u_2, \ldots, u_k \) be vectors in \( \mathbb{R}^n \). Prove that the span of \( \{u_1, u_2, \ldots, u_k\} \) is the same as the span of \( \{u_1 + 2u_2, u_2, \ldots, u_k\} \).

**Sol. of 9:** The span of \( u_1, u_2, \ldots, u_k \) consists of all linear combinations of \( u_1, u_2, \ldots, u_k \), while the span of \( u_1 + 2u_2, u_2, \ldots, u_k \) consists of all linear combinations of \( u_1 + 2u_2, u_2, \ldots, u_k \).

We have to prove that every vector in the first set also belongs to the second set, and vice versa.

It is easier to prove the second statement first. A member of the second set has the following format:
\[
c_1(u_1 + 2u_2) + c_2u_2 + c_3u_3 + \ldots + c_ku_k,
\]
for some \( k \) real numbers \( c_1, c_2, \ldots, c_k \). Using the algebra of vectors (opening-up parentheses and then collecting terms), this equals
\[
c_1u_1 + 2c_1u_2 + c_2u_2 + c_3u_3 + \ldots + c_ku_k = c_1u_1 + (2c_1 + c_2)u_2 + c_3u_3 + \ldots + c_ku_k.
\]
But this is the right format for the first set, since whenever \( c_1, c_2 \) are real numbers, so is \( 2c_1 + c_2 \), so this is a legitimate member of the first set.

In order to prove the first statement we use a trick. Write \( u_1 \) as
\[
u_1 = (u_1 + 2u_2) - 2u_2.
\]
Any linear combination
\[
c_1u_1 + c_2u_2 + \ldots + c_ku_k
\]
can be written as
\[
c_1[(u_1 + 2u_2) - 2u_2] + c_2u_2 + \ldots + c_ku_k = c_1(u_1 + 2u_2) + (c_2 - 2c_1)u_2 + \ldots + c_ku_k,
\]
and the latter is a member of the second set.

Since every vector of the second set also belongs to the first set, and vice versa, this means that these two sets are the same. QED.

**Note:** Very few people got it completely. Quite a few people were on the right track. The above proof is only one correct way.
10. Compute the product of the partitioned matrix using block multiplication.

\[
\begin{bmatrix}
1 & -1 \\
3 & 1 \\
-1 & 5 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 0 \\
-1 & 2 & -1 & 2
\end{bmatrix}
\]

Sol. of 10: We view the matrices as matrices of blocks and give them names:

\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
\]

Where

\[
A_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}
\]

\[
A_2 = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 1 & 2 \end{bmatrix}
\]

\[
B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

\[
B_2 = \begin{bmatrix} 2 & 3 & 0 \\ 2 & -1 & 2 \end{bmatrix}
\]

We first do the matrix-product (of a $2 \times 1$ matrix times a $1 \times 2$ matrix, getting a $2 \times 2$ matrix (symbolically):

\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} =
\begin{bmatrix}
A_1B_1 & A_1B_2 \\
A_2B_1 & A_2B_2
\end{bmatrix}
\]

Now we have to do four matrix-products:

\[
A_1B_1 = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [(1)(1) + (-1)(-1)] = [2]
\]

\[
A_1B_2 = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 2 & -1 & 2 \end{bmatrix} = [0 \ 4 \ -2]
\]

\[
A_2B_1 = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix}
\]

\[
A_2B_2 = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 2 \\ 8 & -8 & 10 \\ 6 & 1 & 4 \end{bmatrix}
\]

Now we place everything on top getting

\[
\begin{bmatrix}
2 & 0 & 4 & -2 \\
2 & 8 & 8 & 2 \\
-6 & 8 & -8 & 10 \\
-1 & 6 & 1 & 4
\end{bmatrix}
\]

This is the answer.