1. (10 pts. altogether) (a) (7 pts) What is the rank of the matrix

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1
\end{bmatrix}
\]

**Sol. to 1a):** You apply the first phase of Gaussian elimination. The elementary row operations \( r_3 - r_1 \to r_3 \) and \( r_4 - 2r_1 \to r_4 \) will get everything below the \((1,1)\) entry to be 0:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1
\end{bmatrix}
\to
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & -1 & 1
\end{bmatrix}
\]

The elementary row operation \( r_4 - r_2 \to r_4 \) will get everything below the \((2,2)\) to be 0:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & -1 & 1
\end{bmatrix}
\to
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Now it is in **row-echelon form**. We see that there are 3 pivots (or equivalently, three rows that are not all-zero). So the rank is 3.

**Ans. to 1(a):** 3.

(b) (3 points) Using part (a) find the nullity of \( A \).

**Sol. to 1(b)):** The nullity is the number of columns \( n \) minus the rank. So it is \( 4 - 3 = 1 \).

**Ans. to 1(b):** 1.
2. (10 pts.) Let 
\[ S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -10 \end{bmatrix} \right\} \]
determine whether the set \( S \) is linearly independent or linearly dependent. In case it is linearly dependent, write the zero vector \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) explicitly as a non-trivial linear combination of the vectors in \( S \).

**Sol. of 2:** This is so simple that we can do it by inspection. The second vector is \(-5\) times the first one, so:
\[ \begin{bmatrix} -5 \\ -10 \end{bmatrix} = (-5) \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]

Moving everything to the left, we get
\[ (5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} -5 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

**Ans. to 2:** \( S \) is linearly dependent and the expression of 0 as a non-trivial linear combination of the vectors of \( S \) is:
\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} -5 \\ -10 \end{bmatrix}. \]

Note: there are many (infinitely many other) ways to do this, for example:
\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (10) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} -5 \\ -10 \end{bmatrix}, \]
and the general way is:
\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (5c) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (c) \begin{bmatrix} -5 \\ -10 \end{bmatrix}, \quad c \neq 0. \]

If you can’t do it by inspection, you form the matrix whose columns are the two vectors
\[ \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \]
and then you can use Gaussian elimination. The elementary row operation \( r_2 - 2r_1 \rightarrow r_2 \) will yield
\[ \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \]
From here you see that the second column is \(-5\) the first column and by the **column-correspondence property** you get the same answer.
3. (10 pts altogether) Let

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
\]

Calculate the following matrix products, if they are defined, or explain why they don’t make sense.

(a) (5 points) \(AB\)

**Sol. to 3a):**

\[
AB = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 & 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 \\
0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 \\
1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 & 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}
\]

(b) (3 points) \(AB^T\)

**Sol. to 3b):** undefined. You can’t multiply a 3 \(\times\) 3 matrix by a 2 \(\times\) 3 matrix.

(c) (2 points) \(C^2\)

**Sol. to 3c):**

\[
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
1 \cdot 1 + 1 \cdot -1 & 1 \cdot 1 + 1 \cdot 1 \\
-1 \cdot 1 + 1 \cdot -1 & -1 \cdot 1 + 1 \cdot 1
\end{bmatrix} = \begin{bmatrix}
0 & 2 \\
-2 & 0
\end{bmatrix}
\].
4. (10 pts.) For the matrix
\[ A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \]
compute the matrix \( A^8 \).

**Sol. of 4):** (corrected Dec. 6, 2010, thanks to Sarita Paul).

\[ A^2 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix} \]
\[ A^4 = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}^2 = \begin{bmatrix} -7 & 4 \\ -8 & -7 \end{bmatrix} \]
\[ A^8 = \begin{bmatrix} -7 & 4 \\ -8 & -7 \end{bmatrix}^2 = \begin{bmatrix} 17 & -56 \\ 112 & 17 \end{bmatrix} \]

**Note:** Some (not too many) people computed \( A^3, A^4, A^5, A^6, A^7, A^8 \). Some even got the correct answer. I had to give these people full credit, since their method is correct, and they got the right answer. But their method is **inefficient**. If they had to compute \( A^{1024} \) with their way, they would need to do 1023 matrix multiplication, whereas with repeated **squaring**, we only need 10 operations.
5. (10 pts.) For the following matrix $A$ finds its reduced-row-echelon form, $R$, and find an invertible matrix $P$ such that $PA = R$.

$$A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 2
\end{bmatrix}$$

**Sol. of 5:** (added Dec. 7, 2010: I thank Sarita Paul for spotting a misprint, the previous last row of $P$ was erroneous)

We first bring the matrix to reduced row echelon form, taking careful note of the elementary row operations:

$$\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}.$$

Now it is in reduced-row-echelon form. This is the first part of the answer, $R$. TO get $P$ we apply the above elementary row operations to the identity matrix:

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
-1 & -1 & 1
\end{bmatrix}.$$

**Ans. to 5:**

$$R = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad P = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
-1 & -1 & 1
\end{bmatrix}$$

**Note:** Since $A$ is not invertible ($R$ is not $I_3$), there are more than one correct $P$. The $R$ is always the same, regardless of the choice of the order of elementary row operations, but the $P$ may be different. That’s why it is good to check that $PA = R$, because there is more than one correct $P$ that makes it come true.
6. (10 pts. altogether) In each case below, give an \( m \times n \) matrix \( R \) in reduced row echelon form satisfying the given condition, or explain why it is impossible to do so.

(a) (4 pts) \( m = 2, n = 3 \) and the equation \( Rx = c \) has a solution for all \( c \).

**Sol. to 6a):** There many “correct solutions”. One of them is:

\[
R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}
\]

**Explanation:** The system \( Rx = c \), in *high-school language* is:

\[
\begin{align*}
x_1 + 2x_3 &= c_1 \\
x_2 + 3x_3 &= c_2
\end{align*}
\]

Obviously you can solve it for any choice of real numbers \( c_1, c_2 \). \( x_3 \) is a free variable, and the general solution is \( x_1 = c_1 - 2x_3, x_2 = c_2 - 3x_3, x_3 = x_3 \), so in this system there are infinitely many solutions for all \( c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \), but this is not what the question demanded, only that there is at least one solution.

(b) (4 pts) \( m = 2, n = 2 \) and the equation \( Rx = c \) has a unique solution for all \( c \).

**Sol. to 6b):**

\[
R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

**Explanation:** The system \( Rx = c \), in *high-school language* is:

\[
\begin{align*}
x_1 &= c_1 \\
x_2 &= c_2
\end{align*}
\]

This system is so simple that it equals its own solution. Obviously there is a unique solution \( x_1 = c_1, x_2 = c_2 \) no matter what \( c_1, c_2 \) are. (There are no free variables, of course).

(c) (2 pts) \( m = 3, n = 3 \) and the equation \( Rx = 0 \) has no solution.

**Sol. to 6c):** impossible. \( x = 0 \) always has a solution, namely 0!
7. (10 pts.) Without first computing $A^{-1}$, find $A^{-1}B$, if

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}
\]

**Sol. of 7:** We perform Gaussian elimination on $A$, keeping track of the elementary row operations

\[
\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Now we mimick the same elementary row operations, in the same order starting with $B$:

\[
\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 3 & -4 \\ -1 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 3 & -4 \\ 1 & -2 & 3 \end{bmatrix}
\]

**Ans. to 7:**

\[
A^{-1}B = \begin{bmatrix} -1 & 3 & -4 \\ 1 & -2 & 3 \end{bmatrix}
\]
8. (10 pts. altogether, 2 each) True or False? Give a short explanation!

(a) For any $n \times n$ matrices $A$ and $B$, if $AB = I_n$, then $BA = I_n$.

**Sol. to 8(a):** True. (theorem)

(b) If $A$ and $B$ are invertible $2 \times 2$ matrices, then so is $A + B$.

**Sol. to 8(b):** False. For example if $A = I_2$ and $B = -I_2$.

(c) The sum of any two $m \times n$ matrices is always defined.

**Sol. to 8(c):** True.

**Note:** Some people answered “False”, since the two matrices “may not have the same $m$ and $n$”. In a different galaxy they may have been right, but the mathematical language in planet Earth (in the Solar System, Milky Way) implies when you say two $m \times n$ matrices, that we are talking about the same $m$ and the same $n$.

(d) The product of any two $4 \times 9$ matrices is never well-defined.

**Sol. to 8(d):** True. For a matrix product to be well-defined the number of columns of the left-matrix must equal the number of rows of the right-matrix.

(e) The equation $Ax = b$ is consistent if and only if $b$ is a linear combination of the rows of $A$.

**Sol. to 8(e):** False. The correct statement is with “rows” replaced by columns.
9. (10 pts.) Let $u$ be a solution of $Ax = b$ and $v$ be a solution of $Ax = 0$, where $A$ is an $m \times n$ matrix and $b$ is a vector in $\mathbb{R}^m$. Show that $u + v$ is a solution of $Ax = b$.

**Sol. of 9:** We are told that $Au = b$, $Av = 0$

Now, by the distributive property and the data of the problem, we have

$$A(u + v) = Au + Av = b + 0.$$  

Since adding 0 does not change the vector, this equals

$$b + 0 = b.$$  

We have just proved that $A(u + v) = b$, but this means that $u + v$ is a solution of $Ax = b$.

**Note:** This question is very abstract, and many people didn’t get it. **But** something very similar was in the Review problems and I posted the answers. I am willing to bet that most of the people who didn’t get it didn’t read the posted answers, that I worked so hard to prepare. Too bad!
10. (10 pts. altogether, 5 each)

(a) What does it mean to say that the vectors $u_1, \ldots, u_k$ in $R^n$ are linearly independent?

**Sol. of 10(a)** $u_1, \ldots, u_k$ are linearly independent if there is no way that the there are $k$ real numbers $c_1, \ldots, c_k$ such that

$$c_1 u_1 + \ldots + c_k u_k = 0$$

unless all of them are equal to 0, i.e. $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

(b) What is meant by the span of a set of vectors $S = \{u_1, \ldots, u_k\}$? Give the precise definition in one or more sentences.

**Sol. of 10b)** the span of $S = \{u_1, \ldots, u_k\}$ is the set of all linear combinations. In other words, it is the set

$$\{c_1 u_1 + \ldots + c_k u_k : -\infty < c_1 < \infty, \ldots, -\infty < c_k < \infty\}.$$