1. (a) What does it mean to say that the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$ are *linearly independent*? Give the precise definition in one or more full sentences.

(b) Are the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ linearly independent? Justify your answer *in terms of the definition you gave in (a).*

2. (a) What is meant by the *span* of a set of vectors $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$? Give the precise definition in one or more full sentences.

(b) Suppose that $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Is the span of the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ all $\mathbb{R}^3$? Justify your answer *in terms of the definition you gave in (a).*

3. In (a)-(c) below we suppose that we have been given a system of equations $A\mathbf{x} = \mathbf{b}$ and that we have already reduced the augmented matrix $[A \ \mathbf{b}]$ to the reduced row-echelon form $[R \ \mathbf{c}]$ given. In each case, determine (i) whether the original equations have a solution; (ii) if they do have a solution, whether or not it is unique; and (iii) if it is not unique, on how many free parameters there are in the solution. Then write the solution explicitly as a fixed vector plus a linear combination of other vectors, with coefficients the free variables.

(a) $[R \ \mathbf{c}] = \begin{bmatrix} 1 & 5 & 0 & 2 & 8 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $[R \ \mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

(c) $[R \ \mathbf{c}] = \begin{bmatrix} 0 & 1 & 2 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. In each part below, give a $m \times n$ matrix $R$ *in reduced row-echelon form* satisfying the given condition, or explain briefly why it is impossible to do so.

(a) $m = 3$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{c}$ has a solution for all $\mathbf{c}$.

(b) $m = 3$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has a unique solution.

(c) $m = 4$, $n = 3$, and the equation $R\mathbf{x} = \mathbf{c}$ has a solution for all $\mathbf{c}$.

(d) $m = 4$, $n = 3$, and the equation $R\mathbf{x} = \mathbf{0}$ has a unique solution.

(e) $m = 4$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has no solution.

(f) $m = 4$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

(g) $m = 4$, $n = 4$, and for every $\mathbf{c}$ the equations $R\mathbf{x} = \mathbf{c}$ have a solution containing a free parameter.

5. (a) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are solutions of the system of equations $A\mathbf{x} = \mathbf{0}$. Show that $c\mathbf{u} + d\mathbf{v}$ is also a solution, for any scalars $c$ and $d$.

(b) Why does the above conclusion not hold (in general) if the system of equations is $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b}$ a nonzero vector?

6. Suppose that

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 \\ 3 & -3 \\ 4 & 1 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 & 6 & 2 \\ -3 & 0 & 4 \end{bmatrix}.$$

Which of the following quantities are defined? Calculate those that are.
(a) $AB$, (b) $AD^T$ (c) $3D - 2B$ (d) $BAC$ (e) $CAB$ (f) $C + 2A$ (g) $C^TC$.

7. Let $A$ be an $m \times n$ matrix of rank $r$. What can you conclude about $m$, $n$, and $r$ (other than $r \le m$ and $r \le n$, always true) if the equation $A\mathbf{x} = \mathbf{b}$ has

(a) exactly one solution for some $\mathbf{b}$ and no solution for other $\mathbf{b}$?

(b) infinitely many solutions for all $\mathbf{b}$?

(c) exactly one solution for every $\mathbf{b}$?

(d) infinitely many solutions for some $b$ and no solutions for other $\mathbf{b}$?

(e) exactly one solution when $\mathbf{b} = \mathbf{0}$?

8. (a) Suppose that $A$ and $B$ are $4 \times 5$ matrices and that $B$ is obtained from $A$ by the row operation given below. In each case give an elementary matrix $E$ such that $B = EA$.

(i) $\mathbf{r}_1 \leftrightarrow \mathbf{r}_4$, (ii) $\mathbf{r}_3 + 3\mathbf{r}_2 \to \mathbf{r}_3$.

(b) Give the inverses of the elementary matrices found in (i) and (ii) above. (You can do this without calculation; think about reversing the corresponding row operations.)

9. A certain $3 \times 3$ matrix $A$ has reduced row echelon form $R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Find explicitly a nontrivial linear relation on the columns of $A$, that is, a relation $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{0}$ with $c_1$, $c_2$, and $c_3$ not all zero.

10. (a) Suppose that $A$ is a square matrix. What does it mean to say that $A$ is *invertible*? (Give the definition, not one of the many equivalent conditions in Theorem 2.6.)

(b) Suppose that $A$ and $B$ are invertible $n \times n$ matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

(c) Suppose that $A$ is an $n \times n$ matrix. A *left inverse* for $A$ is an $n \times n$ matrix $B$ with $BA = I_n$; a *right inverse* for $A$ is an $n \times n$ matrix $C$ with $AC = I_n$. Show that if $A$ has both a right and left inverse then $A$ is invertible and $B = C = A^{-1}$. (Hint: follow the proof from the book or class that the matrix inverse is unique.)

11. Show that the matrix $\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ is invertible, and find its inverse.

12. Let $A = \begin{bmatrix} 1 & -1 & -3 & 4 \\ -2 & 1 & 5 & 0 \\ 4 & -2 & -10 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ -4 \\ 5 \end{bmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination and write the solution explicitly as a fixed vector plus a linear combination of other vectors, with coefficients the free variables.

13. Do all the True-False questions from Sections 1.1–1.4, 1.6, 1.7, 2.1, 2.3, and 2.4.