Solutions to Real Quiz # 3 for Dr. Z.'s MathHistory

1. (3 points) Describe the Dichotomy Paradox (aka half-way paradox).

Ans. to 1: In order to get from your starting point to your destination, you must first visit the half-way point. Having gotten to that half-way point, there is a new one, and now you are $\frac{1}{2}+\frac{1}{4}$ away from the start. And yet another one, and now you are $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ from the start, and so on and so forth. No matter how many previous half-points that you have crossed, there is always another one, hence you will **never** make it to your destination.

2. (3 points) Prove that there are infinitely many triples of positive integers a, b, c such that

$$a^2 + b^2 = c^2 \quad .$$

Sol. to 2: We will show that the 'recipe'

$$a = 2mn \quad , \quad b = m^2 - n^2 \quad , \quad c = m^2 + n^2$$

, produces solutions to $a^2 + b^2 = c^2$. Using $(A-B)^2 = A^2 - 2AB + B^2$ and $(A+B)^2 = A^2 + 2AB + B^2$, we have

$$a^2+b^2-c^2 = (2mn)^2+(m^2-n^2)^2-(m^2+n^2)^2 = 4m^2n^2+m^4-2m^2n^2+n^4-(m^4+2m^2n^2+n^4) =$$

$$4m^2n^2 + m^4 - 2m^2n^2 + n^4 - m^4 - 2m^2n^2 - n^4 = 0 \quad .$$

Whenever m and n are integers, obviously a, b, c are also integers, since there is an infinite supply of integers m and n such that $m > n$ (to make sure that b is positive), we are done.

Comment: What I meant, was to ask you to prove that there are infinitely many **primitive** Pythagorean triples, but I didn't so I gave full cerdit to the people who said that $(3m, 4m, 5m)$ all m are Pythagorean triples.

3. (4 points) Using $a = 2mn, b = m^2 - n^2, c = m^2 + n^2$, find as many Pythagorean triples as you can with $a = 100$. Which ones are **primitive**?

Sol. to 3: Since $100 = 2mn$, we have $mn = 50$. So m and n are divisors of 50, and for b to be positive, $m > n$.

So there are three possibilities

$$m = 50 \quad , \quad n = 1$$

leading to $a = 100$, $b = 50^2 - 1^2 = 2499$ and $c = 50^2 + 1^2 = 2501 \quad .$

$$m = 25 \quad , \quad n = 2$$

leading to $a = 100$, $b = 25^2 - 2^2 = 621$ and $c = 25^2 + 2^2 = 629$.

$$m = 10 \quad , \quad n = 5$$

leading to $a = 100$, $b = 10^2 - 5^2 = 75$ and $c = 10^2 + 5^2 = 125$.

But the last one is **not** primitive, since $(100, 75, 125)$ is a multiple (by 25) of the famous Pythagoream triple $(4, 3, 5)$.

Ans. to 3: There are three Pythagorean triples (a, b, c) with $a = 100$, two of them are primitive:

$$(100, 2499, 2501) \quad , \quad (100, 621, 629)$$

and one is not: $(100, 75, 125)$.