Solutions to the Real Quiz # 2 for Dr. Z.'s MathHistory

1. (2 points) How did the Babylonians approximate square-roots?

Ans. to 1.

$$\sqrt{A} = \sqrt{a^2 + h} = a + \frac{h}{2a} = \frac{1}{2}(a + \frac{A}{a}) \quad .$$

2. (2 points) What approximation to π did the ancient Babylonians use? Is it better or worse than the one used in ancient Egypt?

Ans. to 2.: 3 (much **worse** then $\sqrt{10}$ (ancient India), and $(\frac{16}{9})^2$.

3. (1 point) What was the name of the script used in ancient Babylon, that they wrote on clay tablets?

Ans. to 3.: Cuneiform.

4. (2 points) Construct a four by four magic square.

Sol. to 4. First, fill in, left-to-right, top-to-bottom in order the integers 1 through 16, and enter each in a the appropriate box, **if** it is in the main diagonal or the other diagonal (there are eight of them). If you encounter a box that is not in the above region, you write it down on the side.

Hence of first phase

$$\begin{matrix} 1 & & & 4 \\ & 6 & 7 & \\ & 10 & 11 & \\ 13 & & & 16 \end{matrix}$$

$$2, 3, 5, 8, 9, 12, 14, 15 \quad .$$

Now read the remaining integers from right to left (i.e. in *decreasing order*), and fill in the rest.

End of Second (and final) phase

$$\begin{array}{cccc} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{array} \quad .$$

5. (3 points) Derive, and **prove**, an explicit formula for

$$A(n) := 3 + 7 + 11 + \ldots + 4n - 1 = \sum_{i=1}^{n} (4i-1) \quad .$$

Sol. to 5.

First Part: Discovery!

Since the *summand*, $4i-1$ is of **degree** 1 in the summation variable i, we know *a priori* that $A(n)$ is *some* polynomial of degree 2 in n. So let's write down a **template** for a polynomial of degree 2.

$$A(n) = a + bn + cn^2 \quad ,$$

where a, b, c are **three numbers** to be determined.

We need **three data-points**, to determine a, b, c.

$$A(0) = 0$$

(empty sum) ,

$$A(1) = 3 \quad ,$$

$$A(2) = 3 + 7 = 10 \quad .$$

Hence

$$A(0) = a + b \cdot 0 + c \cdot 0^2 = 0 \quad ,$$

$$A(1) = a + b \cdot 1 + c \cdot 1^2 = a + b + c = 3 \quad ,$$

$$A(2) = a + b \cdot 2 + c \cdot 2^2 = a + 2b + 4c = 10 \quad .$$

We have to solve the following system of three equations for the three unknown numbers a, b, c:

$$a = 0 \quad , \quad a + b + c = 3 \quad , \quad a + 2b + 4c = 10 \quad .$$

Of course, $a = 0$, so we only need to solve

$$b + c = 3 \quad , \quad 2b + 4c = 10 \quad .$$

From the first equation, we have $c = 3 - b$. Plugging into the second, we get

$$2b + 4(3 - b) = 10 \quad .$$

Simplifying

$$2b + 12 - 4b = 10$$

$$2 = 2b \quad .$$

So $b = 1$. Going back to $c = 3 - b$, we get that $c = 2$.

Hence:

Solution to the system of equations

$$a = 0 \quad , \quad b = 1 \quad , \quad c = 2 \quad .$$

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Going back to the template, we have a tentative explicit expression for $A(n)$.

$$A(n) = 0 + 1 \cdot n + 2 \cdot n^2 = n + 2n^2 = n(2n + 1) \quad .$$

Ans. to First Part (discovery phase)

$$\sum_{i=1}^{n} (4i - 1) = n(2n + 1) \quad .$$

Second Part: Proving

Dr. Z.'s style-proof Since the degree of both sides is 2, we only need to check it in three different values. $n = 0$:

$$\sum_{i=1}^{0} (4i - 1) = 0(2 \cdot 9 + 1) = 0 \quad (OK) \quad .$$

$n = 1$:

$$\sum_{i=1}^{1} (4i - 1) = 3 = 1(2 \cdot 1 + 1) = 3 \quad (OK) \quad .$$

$n = 2$:

$$\sum_{i=1}^{2} (4i - 1) = 3 + 7 = 10 = 2(2 \cdot 2 + 1) = 3 \quad (OK) \quad .$$

Hence it is true for every n.

Comment: If you trust that the construction of $A(n)$ in the discovery part of the problem was correct, the second part is 'superfluous', since $A(n)$ was precisely constructed using the facts that $A(0) = 0, A(1) = 3, A(2) = 7$. But we humans always mess up, so it is a good idea to double-check.

Official proof of the discovered identity

$$\sum_{i=1}^{n}(4i-1) = n(2n+1) \quad .$$

The base case $n = 0$ is true since the empty sum is 0, and the right side is $0(2 \cdot 0 + 1) = 0$.

Inductive Hypothesis (replace n by $n-1$)

$$\sum_{i=1}^{n-1}(4i-1) = (n-1)(2(n-1)+1) = (n-1)(2n-1) = 2n^2 - 3n + 1 \quad .$$

Inductive Step (singling out the last term)

$$\sum_{i=1}^{n}(4i-1) = \left(\sum_{i=1}^{n-1}(4i-1)\right) + (4n-1) = 2n^2 - 3n + 1 + 4n - 1 = 2n^2 + n = n(2n+1) \quad .$$

QED!

Comment: In this quiz you did not have to give both styles of proof, but in the exams, I will specify which one I expect (possibly both!). You should be able to do both styles!