Solutions Getting to know you Quiz (does not count towards the grade)

1.: What are your career goals?

They were pretty diverse, medical doctor, software engineer, highschool teacher, data scientist, and others.

2.: What are your hobbies?

They were also diverse. Thanks for sharing

3. What is a rational number?

Sol. A ratio of integers a/b, where $b \neq 0$.

Comment: everyone got it right

4. Prove that the sum of two rational numbers is also a rational number,

Sol.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad ,$$

since the product of two integers is an integer, and the sum of two integers is an integer, and bd is not 0, the right side is also a ratio of integers, hence is a rational number.

Comment: everyone got it right

5. Prove or disprove (by giving a counterexample) : "the sum of two irrational numbers is always also an irrational number"

Sol. It is false. The easiest example is $\sqrt{2} + (-\sqrt{2}) = 0$, and of course 0 is rational. An example where both summands are positive is $(2 - \sqrt{2}) + \sqrt{2} = 0$.

Comment: most people got it right, but two people said that it was right. Please review it.

6. Prove that there are infinitely many primes.

Sol.: Suppose that there are finitely many of them, say k of them, and let's order them, $p_1 < p_2 < \ldots < p_k$ (where $p_1 = 2, p_2 = 3, dots$. let

$$N = p_1 \cdots p_k + 1 \quad .$$

N is not divisible by $p_1, p_2, \ldots, p_k$ (since it leaves remainder 1 always) hence it is either a brand-new prime itself, or divisible by yet-another brand-new prime. Contradiction.

Comment: 4 out of 16 people got it right. Some ran out of time. This is a historically very important proof, please make sure that you understand it.

7. Prove that $\sqrt{5}$ is an irrational number.

Sol. Proof by contradiction. Suppose $\sqrt{5}$ is rational, then there are integers p and q such that

$$\sqrt{5} = \frac{p}{q} \quad .$$

Without loss of generality, p and q have no common factor (i.e. they are *relatively prime*)

Square both sides and simplify, you get

$$p^2 = 5q^2 \quad .$$

Hene p^2 is divisible by 5, hence p is divisible by 5 (since 5 is prime), hence you can write

$$p = 5r \quad ,$$

for some integer r. Hence

$$(5r)^2 = 5q^2 \quad .$$

Simplifying

$$5r^2 = q^2 \quad .$$

Hence q^2 is divisible by 5, hence q itself is, hence you can write $q = 5s$. Hence both p and q have a common factor (namely 5), contradicting the initial assumption.

Comment: About half of the people got it right. If you don't like proofs by contradiction, then instead of making the assumption that p and q have no common factor, think of it as an algorithm that inputs a pair (p, q) with the property $p^2 = 5q^2$ and outputs a smaller pair (r, s) with the same property. Since the smallest pair $(2, 1)$ and it is false for that pair. It also follows.

Another comment: The discovery that there exist irrational numbers is historically very important.