Solutions to Part II of the Attendance Quiz # 15 for Dr. Z.'s MathHistory Lecture 15

1. Prove that in the twenty-four puzzle, if you start with

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & \end{pmatrix}$$

It is **impossible**, by sliding, to get to the position

$$\begin{pmatrix} 2 & 1 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & \end{pmatrix}$$

If you denote the blank space by 16 (or in this problem, 25) and look at the number of inversions of the permutation then the invariant S is

"parity of the number of inversions + row number + column number" modulo 2

(Note: row number + column number minus 2 is called the "taxi-cab distance from the top-left")

If you ignore the 16 (or 25, or in general n^2) and read the permutation without it, then the invariant, let's call it S

"parity of the number of inversions + row number " modulo 2

Solution (using the first, more complicated invariant, but it works for both even and odd sizes):

We first need a lemma that is not trivial, but also not too hard to prove

Lemma: In any permutation, if you exchange any two entries (i.e. the element that was in the j-th place and the element that was in the i-th place trade places, but all the other elements stay in the same place), the difference between their number of inversions is always **odd**.

Consider the permutation obtained by reading it the usual way (as in English), left-to-right and top-to-bottom, **including** the blank space that we label 25 (in general, n^2).

There are two types of legal moves.

(i) Sliding horizontally (staying at the same row), then the parity of the permutation changes (since n^2 got exchanged with one of its neighbors), but also the "taxi cab distance" changes parity (since the column-number changed by one). So the parity of the sum is the same.

(ii) Sliding vertically, then n^2 (alias blank-square) exchanged position with one of its vertical neighbors, and hence by the lemma, the number of inversion changed by an odd integer. But the row-number changed by ± 1, hence the parity of 'number of inversions plus row-number plus column-number' remains the same.

Since any single legal move preserves the parity of S (if it starts being odd, it stays odd, if it starts being even, it remains even), no matter how many moves you make, if it starts out even it remains even for ever after.

Hence it is **impossible** to go from the initial state (that happens to have S even in this problem $0 + 10$) to the state where 1 and 2 are interchanged, since now the parity of S is odd $(1 + 10)$.