Solutions to Attendance Quiz # 12 for Dr. Z.'s MathHistory for Lecture 12

1. Using the definition of the Taylor expansion around $x = 0$ find the first three terms (i.e. $n = 0, 1, 2$) of the function $f(x) = e^{x+x^2}$.

Sol. of 1: The general formula is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \quad ,$$

but we only need to go up to $n = 2$.

We have

$$f(x) = e^{x+x^2} \quad .$$

By the chain rule from calculus

$$f'(x) = e^{x+x^2} \cdot (x + x^2)' = e^{x+x^2} \cdot (1 + 2x) \quad .$$

By the product rule and chain rule

$$f''(x) = (e^{x+x^2} \cdot (1 + 2x))' = (e^{x+x^2})'(1 + 2x) + e^{x+x^2}(1 + 2x)' = e^{x+x^2} \cdot (1 + 2x)^2 + 2e^{x+x^2} \quad .$$

Plugging-in $x = 0$, we get

$$f(0) = e^{0+0^2} = 1 \quad , \quad f'(0) = e^{0+0^2} \cdot (1 + 2 \cdot 0) \quad , \quad f''(0) = e^{0+0^2} \cdot (1 + 2 \cdot 0)^2 + 2e^{0+0^2} = 3 \quad .$$

Hence the first three terms of the Taylor expansion of $f(x) = e^{x+x^2}$ are

$$1 + \frac{1}{1}x + \frac{3}{2}x^2 + \ldots = 1 + x + \frac{3}{2}x^2 + \ldots \quad .$$

Ans. to 1: $e^{x+x^2} = 1 + x + \frac{3}{2}x^2 + \ldots$.

2. Using the famous Taylor expansion (around $x = 0$) of the exponential function, (that you should memorize!), find the first five terms (i.e. $n = 0, 1, 2, 3, 4$, i.e. up to and including the fourth power, x^4) of the function $f(x) = e^{x+x^2}$.

Sol.

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \ldots$$

So

$$e^{x+x^2} = 1 + (x + x^2) + \frac{1}{2}(x + x^2)^2 + \frac{1}{6}(x + x^2)^3 + \frac{1}{24}(x + x^2)^4 + \ldots$$

Since we are **only** interested in powers up-to-and-including x^4 we can safely disregard any higher powers, replacing them by $\ldots$

$$e^{x+x^2} = 1 + (x + x^2) + \frac{1}{2}x^2(1 + x)^2 + \frac{1}{6}x^3(1 + x)^3 + \frac{1}{24}x^4(1 + x)^4 + \ldots$$

$$= 1 + (x + x^2) + \frac{1}{2}x^2(1 + 2x + x^2) + \frac{1}{6}x^3(1 + 3x + \ldots) + \frac{1}{24}x^4(1 + \ldots) + \ldots =$$

$$= 1 + (x + x^2) + \frac{1}{2}x^2 + x^3 + \frac{1}{2}x^4 + \frac{1}{6}x^3 + \frac{3}{6}x^4 + \frac{1}{24}x^4 + \ldots$$

$$= 1 + x + \frac{3}{2}x^2 + (1 + \frac{1}{6})x^3 + (\frac{1}{2} + \frac{1}{2} + \frac{1}{24})x^4 + \ldots = 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{25}{24}x^4 + \ldots \quad .$$

Ans. to 2: $e^{x+x^2} = 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{25}{24}x^4 + \ldots$.