

MATH 437 Solutions to Exam II for Dr. Z.'s, Fall 2021, Dec. 6, 2021 Exam

1. (10 pts.) Give **two** proofs of the Pythagorean theorem.

**Sol.**

**First proof:** Consider an  $(a + b) \times (a + b)$  square, and drawing four right-angled triangles with base  $a$ , height  $b$  and hypotenuse  $c$ , in such a way that each side of the rectangle has one  $a$  and one  $b$ . Then the emerging square at the middle is a  $c \times c$  square, whose area is  $c^2$ , and hence the total area of the large square is  $c^2 + 4 \cdot (ab/2) = c^2 + 2ab$ . On the other hand, the area of the big square is  $(a + b)^2 = a^2 + 2ab + b^2$  so

$$a^2 + 2ab + b^2 = c^2 + 2ab \quad .$$

Subtracting  $2ab$  from both sides of the above equation, we get  $a^2 + b^2 = c^2$ .

**Second Proof:** (Similar triangles). Let  $ABC$  be a right-angled triangle with the angle at  $C$  being the right-angle. Let  $C'$  be the projection of  $C$  on  $AB$ . Then the three triangles

- $ACC'$  (with hypotenuse-size  $|AC| = b$ ),
- $CBC'$  (with hypotenuse-size  $|CB| = a$ ),
- $ABC$  (with hypotenuse-size  $|AB| = c$ ),

are **similar** (they have the same angles), hence there is a non-zero constant  $k$  (whose exact value does not matter) such that the area of triangle  $ACC'$  is  $kb^2$ , the area of triangle  $ABC'$  is  $ka^2$  and the area of triangle  $ABC$  is  $kc^2$ . But of course

$$Area(ACC') + Area(ABC') = Area(ABC) \quad ,$$

so

$$ka^2 + kb^2 = kc^2 \quad .$$

Dividing by  $k$  gives

$$a^2 + b^2 = c^2 \quad .$$

2. (10 pts.) Prove that  $\sqrt[7]{3}$  is irrational.

**Proof:** We need the obvious lemma that for every integer  $N$  all the exponents of  $N^7$  in its prime decomposition are divisible by 7.

Assume that  $\sqrt[7]{3}$  is rational. This means that there exist positive integers  $m$  and  $n$  such that

$$\sqrt[7]{3} = \frac{m}{n} \quad .$$

Raise everything to the seventh-power

$$3 = \frac{m^7}{n^7} \quad .$$

Transposing:

$$m^7 = 3n^7 \quad .$$

The left side has all the exponents of the primes, including the exponent of 3, a multiple of 7, but the exponent of 3 on the right side has the form  $7a + 1$  (i.e. it gives remainder 1 when divided by 7). By the uniqueness of the prime decomposition, this is a contradiction. QED.

**3.** (10 pts. total) (a) (5 points) Construct the Pascal triangle mod 2 Fractal using the first 7 rows (i.e, row 0 through row 7). Highlight the middle 0 section, and show that the remaining part consists of three identical triangles with 4 rows,

**Ans.**

1  
11  
101  
1111  
10001  
110011  
1010101  
1111111

(b) (5 points) Define the Feigenbaum constant. Explain everything!

**Ans.** It turns out that that the mapping from  $[0, 1]$  to itself,  $x \rightarrow rx(1 - x)$  repeated many times, for very small  $r$  ( $r < 1$ ) always go to 0, no matter what the starting number,  $x_0$  is. Later on, until  $r = 3$  it goes to some other fixed point, but only **one**, but starting at  $r_1 = 3$ , for example,  $r = 3.1$ , the limiting behavior vacillates between **two** limiting points. Eventually, at  $r_2$ , the limiting behavior vacillates between **four** limiting points. Soon later at  $r_3$ , the limiting behavior vacillates between **eight** limiting points. It is always a power of 2, and the cut-off places in the parameter  $r$  when it goes from a limiting orbit of  $2^{n-1}$  to a limiting orbit of  $2^n$  are called  $r_n$ .

Michell Feigenbaum proved that

$$\lim_{n \rightarrow \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} ,$$

tends to a fixed constant, about  $4.66 \dots$  and it is called the **Feigenbaum constant**.

4. (10 points altogether)

(a) (2 points) Define a **Platonic solid**

(b) (2 points) Let  $a$  be the number of edges meeting each vertex, and let  $b$  be the number of edges surrounding each face. Express  $V$  (the number of vertices) and  $F$  (the number of faces) in terms of  $E$  (the number of edges), and  $a$  and  $b$ .

(c) (2 points) Find an expressions for  $F$ , in terms of  $a$  and  $b$ .

(d) (4 points) Obviously both  $a$  and  $b$  must be at least 3, and  $F$  (and hence  $V$  and  $E$ ) must be positive. It is easy to see (you don't have to do it) that  $a, b$  must be both between 3 and 5, leaving 9 potential scenarios. Find those values of  $a$  and  $b$  that make sense, and thereby prove that there are exactly 5 Platonic solids. For each of them, find  $F$  (the number of faces) and give the name of the corresponding Platonic solid.

**Sol.**

(a) A platonic solid is a *polyhedron* (a solid body with finitely many faces) where all the faces are identical, and every vertex has the same number of edges coming out of it.

(b) Each of the  $V$  vertices have  $a$  edges meeting it, hence there are altogether  $aV$  edges. But every edge has exactly 2 vertices, so  $aV$  is a **double count** hence

$$aV = 2E \quad .$$

Also every face as  $b$  edges around it, so the total number of edges is  $bF$ . Once again, every edge belongs to exactly two faces, so the above is a double-count. Hence

$$bF = 2E \quad .$$

Hence, using algebra

$$V = \frac{2E}{a} \quad , \quad F = \frac{2E}{b} \quad .$$

(c) Using Euler's famous  $V - E + F = 2$ , we get

$$\frac{2E}{a} - E + \frac{2E}{b} = 2 \quad .$$

Hence

$$E = \frac{2}{\frac{2}{a} - 1 + \frac{2}{b}} = \frac{2ab}{2a + 2b - ab} \quad .$$

We also have (from the above expression,  $F = \frac{2E}{b}$ )

$$F = \frac{4a}{2a + 2b - ab} \quad .$$

Hence we have an expression for  $E$  in terms of  $a$  and  $b$ . Of course  $a \geq 3$  and  $b \geq 3$ . So one has to try out the 9 cases  $a \in \{3, 4, 5\}$ ,  $a \in \{3, 4, 5\}$ . Only  $(a, b) = (3, 3), (a, b) = (3, 4), (a, b) = (4, 3), (a, b) = (3, 5), (a, b) = (5, 3)$  give values of  $E$  that are positive inetgers.

•  $a = 3, b = 3$

$$F = \frac{4(3)}{2(3) + 2(3) - (3)(3)} = \frac{12}{3} = 4$$

This gives the **tetrahedron** (**four** faces).

- $a = 3, b = 4$

$$F = \frac{4(3)}{2(3) + 2(4) - (3)(4)} = \frac{12}{2} = 6$$

This gives the **hexahedron** (**six** faces), more commonly called the **cube**.

- $a = 3, b = 5$

$$F = \frac{4(3)}{2(3) + 2(5) - (3)(5)} = \frac{12}{1} = 12$$

This gives the **dodacahedron** (**twelve** faces).

- $a = 4, b = 3$

$$F = \frac{4(4)}{2(4) + 2(3) - (4)(3)} = \frac{16}{2} = 8$$

This gives the **octahedron** (**eight** faces).

- $a = 5, b = 3$

$$F = \frac{4(5)}{2(5) + 2(3) - (5)(3)} = \frac{20}{1} = 20$$

This gives the **icasahedron** (**twenty** faces).

## 5. (10 points)

Prove Lagrange's theorem that if  $H$  is any subgroup of a group  $G$ , and  $|H|$  and  $|G|$  are their number of elements, respectively, then  $|G|/|H|$  is always an integer.

**Sol.** Let the number of elements of  $G$  be  $n$  and the number of elements of  $H$  be  $m$ .

Let's write

$$H = \{h_1, \dots, h_m\} \quad ,$$

where, of course,  $h_1, h_2, \dots, h_m$  are all **different**.

If  $G = H$  then we are done and  $n/m = 1$ .

Otherwise, there exist a  $g_1 \in G$  **not** in  $H$ . Consider the set (called "coset")

$$g_1H = \{g_1h_1, \dots, g_1h_m\} \quad ,$$

The elements of  $g_1H$  are all **different**, since if not there would have been for some  $1 \leq i < j \leq m$   $g_1h_i = g_1h_j$ . Multiplying both sides, from the left by  $g_1^{-1}$  would give  $h_i = h_j$ , a contradiction.

Also  $g_1H$  and  $H$  have no elements in common. Suppose that they did, then there is an  $h_i$  and  $h_j$ , both in  $H$  such that

$$g_1h_i = h_j \quad .$$

Multiplying both sides, from the right by  $h_i^{-1}$  gives

$$g_1h_ih_i^{-1} = h_jh_i^{-1} \quad .$$

So

$$g_1 = h_j h_i^{-1} \quad .$$

Since  $H$  is a group,  $g_1 \in H$ , a contradiction. If  $G = H \cup g_1 H$  then we are done. Otherwise we can find  $g_2 \in G$  such that  $g_2 \notin H$  and  $g_2 \notin g_1 H$ . By the same argument the elements of  $g_2 H$  are all distinct, and they have no overlap with  $H$ , and  $g_1 H$ . Continuing this way, finally we arrive at  $g_1, \dots, g_{r-1}$  such that

$$G = H \cup g_1 H \cup g_2 H \cup \dots \cup g_{r-1} H \quad ,$$

each of them has  $m$  elements, and they don't overlap, so  $n = mr$ , hence  $r = n/m$  is an integer.

**6.** (10 points) What is the name of the following famous equation-pair?

$$u_x = v_y \quad , \quad u_y = -v_x \quad ,$$

or, in fuller notation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad .$$

What is special about the function  $u(x, y) + iv(x, y)$  where  $u(x, y), v(x, y)$  satisfy the above system of two equations?

**Ans.** The Cauchy-Riemann Equations.  $u(x, y) + iv(x, y)$  is a (complex)-analytic function, i.e. of the form  $F(x + iy)$ .

**Comment:** As pointed out by Sarah C., this material was in the part of the book that I did not ask you to read. People who got this problem right anyway, would get the full ten points, but anyone else, this problem would not count. I will add-up your score, divide by 110 and multiply by 120.

**7.** (10 points) Who discovered the quaternions? What city did that person live in?

**Ans.** William Rowan Hamilton. Dublin.

**8.** (10 points) What is Heron's formula, what century did Heron live in?

**Ans.**  $A = \sqrt{s(s-a)(s-b)(s-c)}$ , where  $A$  is the area of a triangle, and  $a, b, c$  are the lengths of its sides. First-century AD.

**9.** (10 points) Where did Isaac Newton study? Who was his teacher? What unusual action did that teacher do? What was Newton's position after he left Cambridge?

**Ans.** Cambridge University, England. Barrow. Barrow gave up his professorship so that Newton can have it. Master of the Mint.

10. (10 points) In what city was Leibnitz born? Where did he spend most of his life? What King of England was once the employer of Leibnitz?.

**Ans.** Leipzig. Court of Hanover. George I.

11. (10 points total)

(a) (5 points) State Viète's infinite product for  $\frac{2}{\pi}$ .

**Ans.**

See wikipedia.

(b) ( 5 points) State the names of two people who initiated the use of logarithms

**Ans.** Napier and Briggs.

12. (10 points altogether) (a) (3 points) Define a *Eulerian path* in a graph.

(b) (3 points) State the necessary condition for a graph to have a Eulerian path

(c) (4 points) Prove (or explain in your own words) why the condition in (b) is necessary.

**Sol.** (a) A Eulerian path in a graph is a path that starts at some vertex and ends at either the same vertex or a different vertex, and such that every edge is visited **exactly** once.

(b) Every vertex must have an even number of edges coming out of it, except two of them. The two exceptions, signal the starting and ending vertices (or vice-versa).

(c) If such a path exists, except for the first and last vertices, whenever you enter a vertex, you must leave it. Since one is not allowed to use the same edge twice, the number of edges incident to any such vertex (except the first and the last) is even, since they come in pairs. If the first and last vertices are the same (then it is called a **Eulerian cycle**), then these two also have even degrees, otherwise the starting and ending vertices have both odd degrees.