Solutions to MATH 437 Exam I for Dr. Z.'s Math History Course Fall 2021, Oct. 27, 2021

1. (10 pts.) Prove that there are infinitely many primes.

Sol. Suppose that there are only finitely many of them, say k of them, and let's call them $p_1, \dots, p_k$. Let's create a brand-new integer

$$P = p_1 \cdot p_2 \cdots p_k + 1 \quad .$$

P, being a positive integer, must be either a prime itself, or divisibile by at least one prime.

Note that

• It is **not divisible** by p_1, since when you divide P by p_1 you get remainder 1

• It is **not divisible** by p_2, since when you divide P by p_2 you get remainder 1

...

• It is **not divisible** by p_k, since when you divide P by p_k you get remainder 1

So it must be divisible (or be itself) by yet another prime **none of the above**. So we found another prime! This contradicts the assumption that $p_1, \dots, p_k$ are the **only** primes in the world. So whenever you think that you have found all the primes, you can always come up with yet-another-one, hence there are infinitely many of them.

2. (10 pts.) Prove that $\sqrt{29}$ is irrational.

Sol.

We first prove a

Lemma: If n^2 is divisible by 29 then also n must be divisible by 29.

Proof of Lemma: By the **fundamental theorem of arithmetics** any positive integer can be written (uniquely) as a product of prime powers

$$n = p_1^{m_1} \dots p_k^{m_k} \quad .$$

Hence, squaring

$$n^2 = p_1^{2\,m_1} \dots p_k^{2\,m_k} \quad .$$

If 29 were not divisible by n^2, then obviously 29 can not show up in the prime decomposition of n^2, so if it *does* show up then (it has an even exponent) and it must show up in the prime decomposition of n. Hence n is divisible by 29.

Proof that $\sqrt{29}$ is irrational:

Suppose, for the sake of argument, that $\sqrt{29}$ can be written as

$$\sqrt{29} = \frac{m}{n} \quad ,$$

where m and n are both positive integers. If m and n are both divisible by 29, we can cancel out 29 until at least one of them is not divisible by 29.

So if there exists a pair of positive integers m and n such that $\sqrt{29} = \frac{m}{n}$, then there also exists a pair of integers (let's call them again m and n) such that $\sqrt{29} = \frac{m}{n}$, and m and n are **not both divisible by 29**.

Squaring both sides

$$29 = \frac{m^2}{n^2} \quad .$$

By algebra

$$m^2 = 29n^2 \quad .$$

Hence m^2 is divisible by 29, it follows from the lemma that m is divisible by 29, hence we can write

$$m = 29a \quad ,$$

for *some* positive integer a.

Hence

$$(29\,a)^2 = 29n^2 \quad .$$

By algebra

$$29^2\,a^2 = 29\,n^2 \quad ,$$

More algebra

$$n^2 = 29\,a^2 \quad ,$$

hence, n^2 is a multiple of 29, and by the lemma, also n is divisible by 29. So *both* m and n are divisibile by 29, contradictiong the assumption that m are **not** both divisible by 29. Hence we have to renounce the assertion that $\sqrt{29}$ can be written as $\frac{m}{n}$ for positive integers m and n. This means that $\sqrt{29}$ is **irrational**.

3. (10 pts) Derive (from scratch, only using geometric series and calculus) the Taylor series around $x = 0$ of the function

$$\arctan x^3 \quad .$$

Explain!

Sol.

By calculus, $(\arctan x)' = \frac{1}{1+x^2}$, hence

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt \quad .$$

Recall the famous *infinite geometric series* (valid for $|w| < 1$)

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad .$$

Plugging-in $w = -t^2$, we get

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n} \quad .$$

Integrating, *term-by-term*

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{t^{2n+1}}{2n+1} \right) \Big|_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad .$$

Finally we replace x by x^3 getting

Final answer:

$$\arctan\, x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3\,(2n+1)}}{2n+1} \quad .$$

4. (10 pts. altogether) Prove that

$$\sum_{k=0}^{n} k(k-1)(k-2) = \frac{(n+1)n(n-1)(n-2)}{4} \quad ,$$

(i) (5 points): The Dr. Z. way (verifying it for sufficiently many special cases, explain how many you need)

The **summand**, $k(k-1)(k-2)$, is a polynomial of degree 3, hence the sum on the left side is a polynomial of degree 4. The right side is also a polynomial of degree 4, hence to prove that both sides are **always** the same (i.e. for **every** positive integer), it suffices to check 5 **different** special cases. The easiest are $n = 0, 1, 2, 3, 4$.

Calling the left side $L(n)$, and the right side, $R(n)$, obviously

$$L(0) = R(0)(=0) \quad , \quad L(1) = R(1)(=0) \quad , \quad L(2) = R(2)(=0) \quad ,$$

Now

$$L(3) = 1 \cdot 2 \cdot 3 = 6 \quad , \quad R(3) = \frac{(3+1) \cdot 3 \cdot 2 \cdot 1}{4} = 6 \quad ,$$

$$L(4) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 = 30 \quad , \quad R(4) = \frac{(4+1) \cdot 4 \cdot 2 \cdot 2}{4} = 30 \quad .$$

Since $L(n) = R(n)$ for the five **different** arguments $n = 0$, $n = 1$, $n = 2$, $n = 3$, $n = 4$, it follows that $L(n) = R(n)$ for **all** non-negative integers n

(ii) (5 points): The traditional way, using **complete mathematical induction**.

The **base case** $n = 0$ is obviously true.

Let's call the statement $S(n)$. We need to prove for **all** integers $n \geq 0$,

So we **have to prove** the statement $S(n)$, namely:

$$\sum_{k=0}^{n} k(k-1)(k-2) = \frac{(n+1)n(n-1)(n-2)}{4} \quad .$$

$S(0)$ is obviously true (both sides are 0), $S(n)$ would follow for **all** $n \geq 0$, if we can prove

$$S(n-1) \quad IMPLIES \quad S(n) \quad .$$

$S(n-1)$ is called the **inductive hypothesis**. It is obtained by doing a 'global replace' of n by $n-1$. In other words it is

$$\sum_{k=0}^{n-1} k(k-1)(k-2) = \frac{n(n-1)(n-2)(n-3)}{4} \quad .$$

Next we look at the left side of $S(n)$ and 'pull out' the last term

$$L(n) \ = \ \sum_{k=0}^{n} k(k-1)(k-2) \ = \ \left(\sum_{k=0}^{n-1} k(k-1)(k-2) \right) + n(n-1)(n-2) \quad .$$

By the inductive hypothesis this equals

$$\frac{n(n-1)(n-2)(n-3)}{4} + n(n-1)(n-2) = n(n-1)(n-2) \left(\frac{n-3}{4} + 1 \right)$$

$$= n(n-1)(n-2) \left(\frac{n-3+4}{4} \right) = n(n-1)(n-2) \frac{n+1}{4} \quad ,$$

but this is exactly the right side of $S(n)$. **QED**.

5. (10 points) Construct a seven by seven Magic Square.

Ans. to 5:

4	35	10	41	16	47	22
29	11	42	17	48	23	5
12	36	18	49	24	6	30
37	19	43	25	7	31	13
20	44	26	1	32	14	38
45	27	2	33	8	39	21
28	3	34	9	40	15	46

6. (10 points) Arrange the following people according to their year-of-birth, from oldest to youngest.

Newton, Archimedes, Gallileo, Euler, Gauss, Zeilberger, Euclid, Thales, Brahmagupta, Fibonacci.

For each person, state their century of birth.

Ans. to 6:

Thales: sixth century BC

Euclid: fourth century BC

Archimedes: third century BC (more precisely 287 BC)

Brahmagupta: seventh century

Fibonacci: late 12th century

Galilleo: late 16th

Newton: 17th century (1642)

Euler: 18th century

Gauss: late 18th

Zeilberger: 20th century (more precisely, July 2, 1950).

7. (10 points). What is an Egyptian fraction? Express $\frac{5}{6}$ as an Egyptian fraction

Ans. to 7: Expressing a fraction as a sum of **different** unit fractions (pure reciprocals). $\frac{1}{2} + \frac{1}{3}$.

8. (10 points) What is the difference between Ionian (Greek) mathematics and ancient Babylonian and Chinese mathematics? Who was the traditional father of Greek mathematics?

Ans. to 8: The former was *pure* the latter was 'applied', practical, and not proof-based. Thales of Milete.

9. (10 points) What book, except for the bible, was the most reproduced and studied in the Western world? Who was its author?

Ans. to 9: 'The Elements', by Euclid.

10. (10 points) In a closed polyhedron, what is a relation between V, the number of vertices, E, the number of edges, and F, the number of faces? Who is it due to?

Ans. to 10: $V - E + F = 2$. It is due to Euler.

11. (10 points) What is the name, of the following constant:

$$\lim_{n \to \infty} \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \right) \quad .$$

What is its approximate value?

Ans. to 11: Euler's constnant $\gamma = 0.57721\ldots$.

12. (10 points) Using the beginning of the famous Taylor expansion, about $x = 0$ for $\sin x$, namely

$$\sin(x) = x - \frac{1}{6} x^3 + \ldots \quad ,$$

find the beginning (up to term x^3) of the Taylor series, about $x = 0$ of

$$f(x) = \sin \sin \sin x \quad ,$$

in the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x_3 + \ldots$$

Sol. of 12:

We first need the first few terms (up to and including x^3) of $\sin \sin x$

$$\sin \sin x = \sin(x - \frac{1}{6}x^3 + \ldots) = (x - \frac{1}{6}x^3 + \ldots) - \frac{1}{6}\,(x - \frac{1}{6}x^3 + \ldots)^3$$

$$= x - \frac{1}{6}x^3 - \frac{1}{6}\,x^3 + \ldots = x - \frac{1}{3}x^3 + \ldots$$

(since we can safely discard powers higher than x^3). Finally

$$\sin \sin \sin\, x = \sin(\sin \sin\, x) = (\sin \sin x) - \frac{1}{6}(\sin \sin x)^3 + \ldots \quad .$$

Using what we know so far this is

$$(x - \frac{1}{3}x^3) - \frac{1}{6}(x - \frac{1}{3}x^3)^3 + \ldots$$

$$= x - \frac{1}{3}x^3 - \frac{1}{6}x^3 + \dots \quad .$$

(since we can safely discard powers higher than x^3), and this equals

$$x - \frac{1}{2}x^3 + \dots \quad .$$

Ans.:

$$a_0 = 0; \qquad a_1 = 1; \qquad a_2 = 0; \qquad a_3 = -\frac{1}{2}.$$