

MAGIC SQUARES

By Jonathan Dimond

An ancient Chinese legend says that a turtle that crawled from the River Loh bore a special design on its back. This design had a matrix of squares with numbers within each, whose rows, columns and diagonals added up to the same total.

4	9	2
3	5	7
8	1	6

Odd-number magic squares

This magic square is a typical 3-by-3 magic square, and contains nine different numbers (one of each of 1 to 9). Nine being an odd number, we categorize the 3-by-3 magic squares with other odd number magic squares. The rows, columns and diagonals in this example add up to 15.

15 is considered the *magic number* of this square. It can be deduced quickly by multiplying the central number by 3. I.e. $5 \times 3 = 15$.

ROWS, COLUMNS AND DIAGONALS ADD UP TO THE MAGIC NUMBER

MAGIC NUMBER = CENTRAL NUMBER X 3

If you write out the sequence of numbers in order, you will find that the number that occupies the middle position in the square is also the middle number in the sequence.

1, 2, 3, 4, 5, 6, 7, 8, 9

Try adding together these numbers to find the total summed value of the magic square's numbers. $1+2+3+\dots+9=45$.

The relationship of the middle number to the total summed value of the magic square is always:

TOTAL SUM OF MAGIC SQUARE'S NUMBERS = CENTRAL NUMBER X 9

There are eight possible magic squares of this type. Try to discover the other seven, using only the numbers in the above list. Remember that the rows, columns and diagonals must add up to 15. Here is another one to start you off:

4	3	8
9	5	1
2	7	6

Gave up? Read on and learn some shortcuts that are better than trial-and-error!

- Write the middle number in the sequence (5) in the middle position of the 3-by-3 square.
- Place the first and last numbers in the sequence (1 and 9) diametrically opposite

each other.

- Check that they add up to the magic number (15).
- Proceed by placing the next-to-first and second-last numbers in the sequence (2 and 8) diametrically opposite each other.
- Check that they add up to the magic number (15).
- Continue similarly until all squares are filled in.

Rather than using consecutive numbers, try using a series that advances by a specific interval, for example:

By 2's: 2, 4, 6, 8, 10, 12, 14, 16, 18

By 3's: 3, 6, 9, 12, 15, 18, 21, 24, 27

Again, write the middle number in the sequence (emphasized in these examples) in the middle position in the 3-by-3 square, and follow the dot points (above).

Below is one example built on the first sequence (by 2's). Its magic number is 30, and the total summed value is 90.

12	2	16
14	10	6
4	18	8

The next size up from a 3-by-3 magic square in this category is a 5-by-5 magic square. Now having five rows and columns means that the magic number is found by multiplying the central number by 5.

MAGIC NUMBER = CENTRAL NUMBER X 5

The example below uses all the numbers from 1 to 25, with the middle number of this sequence (13) again appearing in the central position. For the magic number we find $13 \times 5 = 65$.

1	24	17	15	8
14	7	5	23	16
22	20	13	6	4
10	3	21	19	12
18	11	9	2	25

Now that we have a longer sequence of numbers, instead of writing them all out and finding the middle number we can apply the simple formula for finding the middle number in an odd-number sequence:

$$\text{CENTRAL NUMBER} = \frac{(\text{TOTAL NUMBER} + 1)}{2}$$

In the above example, $25 + 1 = 26$. $26 / 2 = 13$.

To create the above example, I followed the same dot points as listed previously for filling

Even-number magic squares

The smallest even-number magic square is a 4-by-4. A partly-completed example appears below:

1			4
	6	7	
	10	11	
13			16

It is easy to see that the numbers that have been filled in are simply the ordered sequence of counting numbers from 1 to 16, reading from left to right and down the rows. Can this be the beginning of a magic square?

It can, but what is the magic number? Obviously, there is no “middle” number in an even square. Adding up the numbers in the two completed diagonals both yield 34. Some other possible formulae and patterns: the sum of the diagonally-opposite corners both yield 17 ($1+16 = 13+4 = 17$); The sum of the largest and smallest numbers on the square is 17 ($16+1=17$); There are 16 possible numbers to be used. That’s one less than 17, too! Let’s try this theorem, then:

MAGIC NUMBER = 2 X (Largest + Smallest Numbers)

Try now completing the square above with the remaining numbers.
2, 3, 5, 8, 9, 12, 14, 15

Look below when you have an answer. (Don’t give up!)

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

There are many 4-by-4 magic squares to discover. Try another using only the numbers 1 through 16.

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16

Analyze my example below. Do you see similarities with the first 4-by-4 square? I began with the same pattern of eight numbers given on the previous page, and then substituted each of the middle four with a different number. The substituted number is

always the difference between the given one and 17 (the sum of the largest and smallest numbers in the series)! I.e. $17-6=11$ and $17-7=10$, etc.

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

Alphabet magic squares

By assigning each number to a letter, you can code a magic square with words or musical notes. At the same time, let's dispense with the rule that each number can only be used once, and fill a 4-by-4 magic square with only the numbers 1 to 4:

1	4	2	3
2	3	1	4
3	2	4	1
4	1	3	2

Each row, column and diagonal uses each of the numbers 1 through 4 once, and thus add up to 10 ($1+2+3+4 = 10$).

Now let's define the letters H E L P as being these four numbers. The result:

H	P	E	L
E	L	H	P
L	E	P	H
P	H	L	E

Perhaps you can see the permutations more easily with letters than numbers?

H	P	E	L
E	L	H	P
L	E	P	H
P	H	L	E

Musical applications of magic squares

Replacing numbers with notes from the musical alphabet would be one obvious application of magic squares to music. However, where the real potential of magic squares lies is in the area of rhythm.

The unity and logic of magic squares must have attracted Indian musicians to applying them to their music, and they certainly seem perfect for applying to time cycles and additive rhythm. Take for example the following 3-by-3 magic square:

3	5	7
5	8	11
7	11	15

In this kind of magic square where consecutive counting numbers are not used, the magic number rule does not apply. You will also notice that all the rows and columns do not add up to the same number. However, the same relationship of the middle number to the total summed value of the magic square applies:

TOTAL SUM OF MAGIC SQUARE'S NUMBERS = CENTRAL NUMBER X 9

This is musically important, as this formula will tell us the size of the magic square, i.e. how many pulses or subdivisions there are in the whole sequence. This will therefore give us ideas on how and where to apply it - in what metre, starting and ending at what points, etc.

In our first example, $8 \times 9 = 72$. This is the magic square's size, and we could apply it to a passage lasting 18 beats in 16th notes. ($18 \times 4 = 72$)

Before we improve on the application of this square, let's examine the numbers further. How is this a magic square anyway if the rows and columns do not add up to the same number? Firstly, the numbers are generally small, and this makes musical sense because larger numbers normally need to be broken into smaller groups and so lose their feeling of wholeness. Secondly, the sequence has a neat internal logic, where the first row increments by 2's, the second by 3's, and the third by 4's. You will also see the pattern where the first three numbers (3, 5, 7) become the starting numbers for each row (top, middle, bottom). The result of these two facets gives an increasing pattern of 9 per row - that is, the three rows sum to 15, 24 and 33. The fact that there are three rows and three numbers in each also gives it a self-similar tripartite structure - much like an Indian *tihai* (*moohra*) or *chakradar tihai*. A tripartite structure makes for a great rhythmic *cadence*, which naturally gravitates towards the very next beat after the end of third repetition. Look at the final number in this magic square, 15. Itself can be subdivided into three parts (5+5+5)!

Try counting, in a continuous stream of subdivisions, all the numbers in this square in order (from left to right and top to bottom). Clap the first number of each. Is the pattern audible?

Now although generally small, these numbers lend themselves to some internal grouping. This will make the pattern more audible. A typical Indian method would be to have a unity of number of subgroups in each number. Take the first row:

$$3 = 1+1+1$$

$$5 = 2+2+1$$

$$7 = 3+2+2$$

Each has three attacks or "hits". Continue with the second row:

$$5 = 1+1+1+1+1$$

$$7=2+2+1+1+1$$

$$11=4+4+1+1+1$$

Each has five attacks or "hits". Continue with the third row:

$$7=1+1+1+1+1+1+1$$

$$11=3+3+1+1+1+1+1$$

$$15=5+5+1+1+1+1+1$$

Each has seven attacks or "hits".

So I have continued the self-similarity of this square by using the first number in each row to define the number of attacks in every member of that row! Furthermore, the decelerating feeling which you got when you first clapped and counted out this magic square is still intact, as you will see that the size of the first grouping in each successive number in each row (above) is increasing:

1, 2, 3 then 1, 2, 4, and lastly 1, 3, 5.

The musical application of this size 72 magic square does not end with passages lasting 18 beats in 16th notes. What is typically done is "gaps" or rests/spaces are added in between the first and second and second and third rows. This doesn't destroy the feeling of the row's unity, but rather can add to the building feeling of this cadence. By choosing different values for the "gap" you can adapt one magic square into many musical situations!

Take for example the same square with (two) gaps of 12.

$72 + (2 \times 12) = 96$. A 96 magic square would fill 48 beats (or 12 bars of 4/4) in 8th notes. Or 32 beats (8 bars of 4/4) in 8th note-triplets. Very "normal"! Here's how the square looks with gaps of 12:

3	5	7 (12)
5	8	11 (12)
7	11	15

It is normal to articulate (attack) the first note in the two gaps. This makes the first note immediately following the magic square feel like the third big attack of the series. I.e. it will strongly feel like the cadence's destination.

Let's try another magic row of this kind.

3	5	7
7	9	11
11	13	15

This one starts out the same way as the first example, but then retains an increment of two per number, starting on the last number of the previous row. The central number (9) times 9 yields the total value for this square.

$$9 \times 9 = 81.$$

And one more example with the same first row but with each successive row starting on the second-last number and incrementing by the same amount appears below:

3	5	7
5	7	9
7	9	11

The central number (7) times 9 yields the total value for this square.
 $7 \times 9 = 63$.

So these are the three musical types of magic square in 3x3 format, to summarize:
 TYPE ONE:

A	B	C
B	D	E
C	E	F

Whereby:

- D-C yields the extra amount by which both rows two and three increment, compared to the previous row.

TYPE TWO:

A	B	C
C	D	E
E	F	G

Whereby:

- All numbers increment by the same amount (x) so that $G = A + (6x)$

TYPE THREE:

A	B	C
B	C	D
C	D	E

Whereby:

- All numbers increment by the same amount (x) so that $E = A + (4x)$

Solutions

The eight 3-by-3 Magic Squares (with counting numbers of 1 through 9)

2	7	6
9	5	1
4	3	8

2	9	4
7	5	3
6	1	8

4	3	8
9	5	1
2	7	6

4	9	2
3	5	7
8	1	6

6	7	8
1	5	9
8	3	4

6	1	8
7	5	3
2	9	4

8	3	4
1	5	9
6	7	2

8	1	6
3	5	7
4	9	2