Real Quiz 9

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1. Prove that every simple planar graph is 5-colorable.

Proof. The proof is similar to that of Theorem 17.3, although the details are more complicated. We prove the theorem by induction on the number of vertices, the result being trivial for simple planar graphs with fewer than six vertices. Suppose then that G is a simple planar graph with n vertices, and that all simple planar graphs with $n-1$ vertices are 5-colourable. By Theorem 13.6, G contains a vertex v of degree at most 5. As before, the deletion of v leaves a graph with $n-1$ vertices, which is thus 5-colourable. Our aim is to colour v with one of the five colours, so completing the 5-colouring of G.

If $\deg(v) < 5$, then v can be coloured with any colour not assumed by the (at most four) vertices adjacent to v, completing the proof in this case. We thus suppose that $\deg(v) = 5$, and that the vertices $v_1, \ldots, v_5$ adjacent to v are arranged around v in clockwise order as in Fig. 17.4. If the vertices $v_1, \ldots, v_5$ are all mutually adjacent, then G contains the non-planar graph K_5 as a subgraph, which is impossible. So at least two of the vertices v_i (say, v_1 and v_3) are not adjacent.

We now contract the two edges vv_1 and vv_3. The resulting graph is a planar graph with fewer than n vertices, and is thus 5-colourable. We now reinstate the two edges, giving both v_1 and v_3 the colour originally assigned to v. A 5-colouring of G is then obtained by colouring v with a colour different from the (at most four) colours assigned to the vertices v_i. //