Hamiltonian Lecture

8-22.

Theorem (Ore, 1960)

G simple with $n (\geq 3)$ vertices.

If

(*) $\deg(w) + \deg(v) \geq n$ for all non-adj pairs $v, w \in V(G)$,

then

G is Hamiltonian.

Proof (Contradiction)

Suppose that G satisfies (*) but is not Hamiltonian.

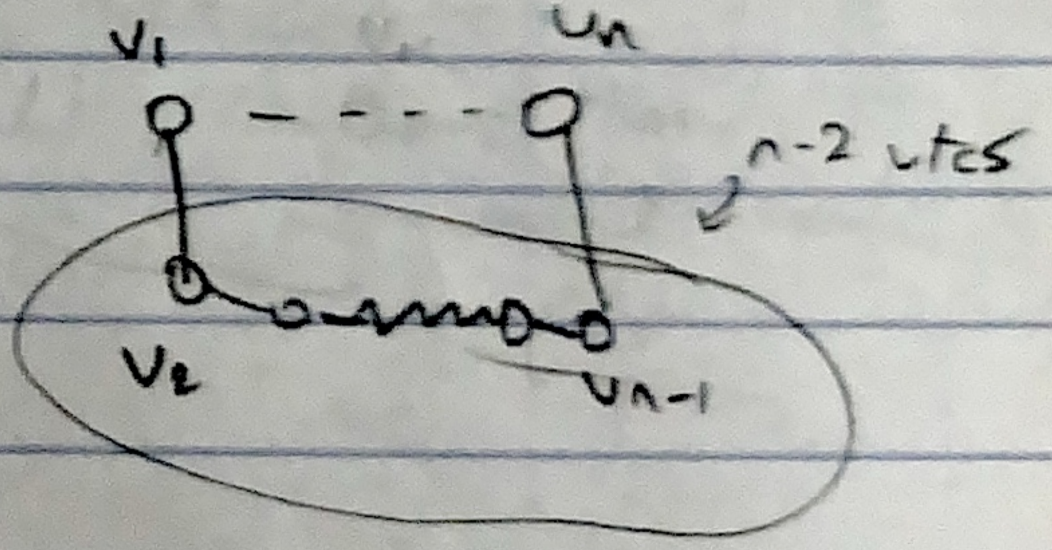
Then, there is a "maximal counter-example": (add an edge if it does not make G Hamiltonian & repeat until you can add no more edges).

WMA G is maximal.

Label the Hamiltonian path:

$v_1 \ldots v_n$

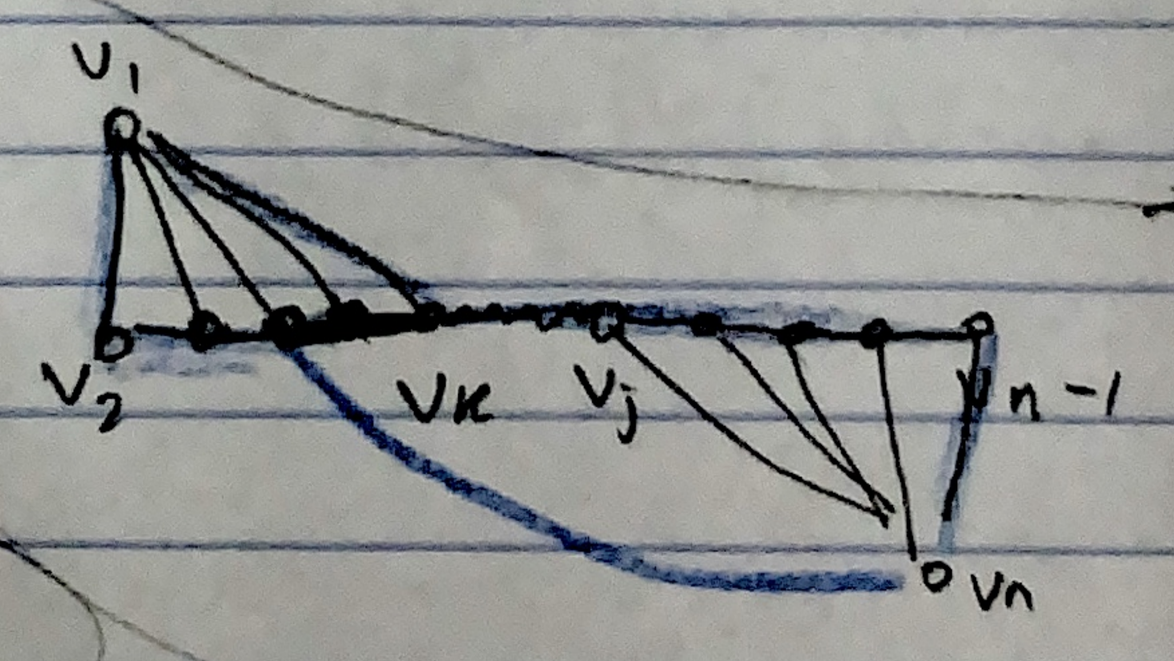
and observe $v_1 v_n \notin E(G)$.



Let $k := \max$ s.t. $\{v_2, \ldots, v_k\} \subseteq N(v_1)$

$j := \min$ s.t. $\{v_j, \ldots, v_{n-1}\} \subseteq N(v_n)$.

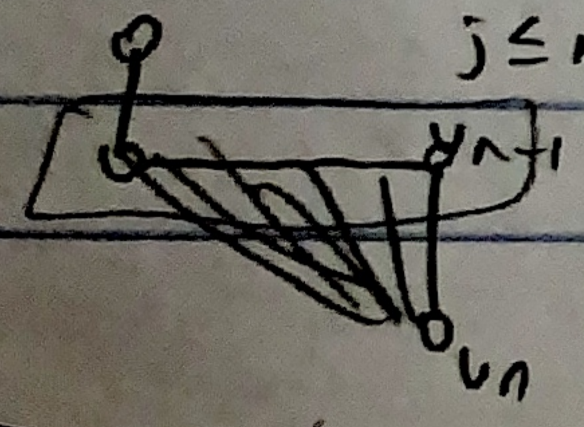
Picture:



Obs.: v_n has no neighbors in $\{v_2, \ldots, v_{k-1}\}$; otherwise ....

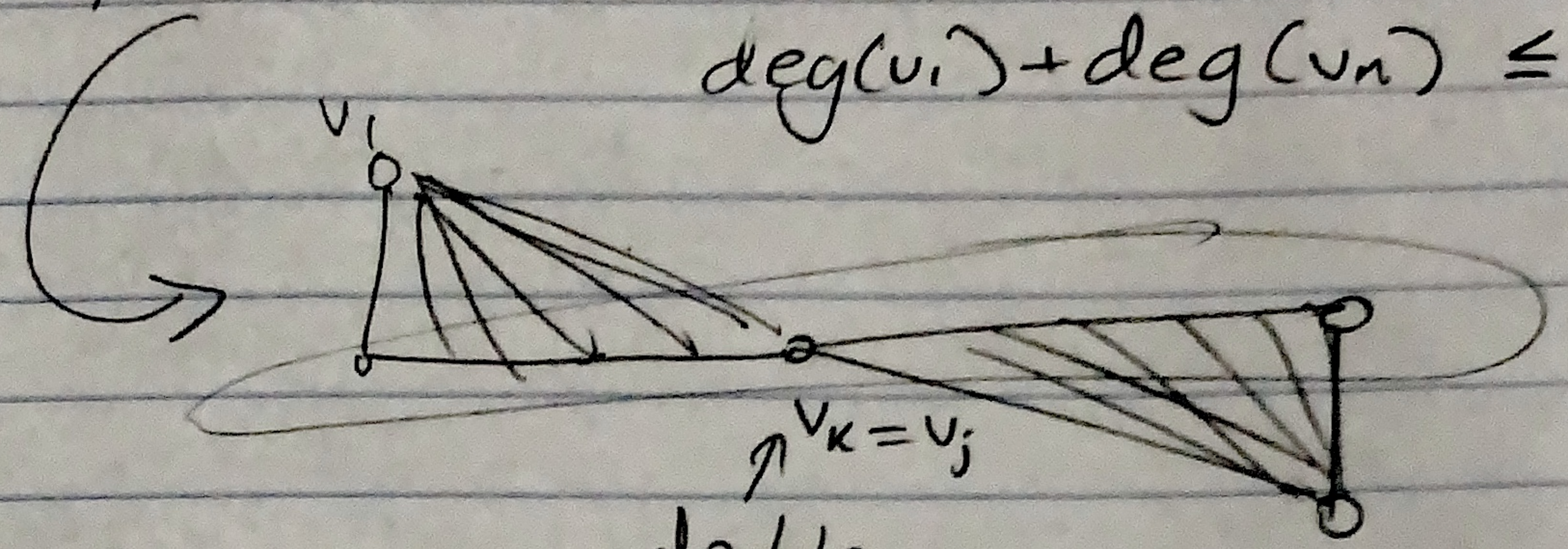
Similarly, v_1 has no neighbors in $\{v_{j+1}, \ldots, v_{n-1}\}$.

Note: $k \geq 3$, and $j \leq n-2$



best case.

However, even in the best case
$$\deg(v_1)+\deg(v_n) \le n-1$$



$n-2$ vtcs

violating $\circledast$ $\times$.

<u>Cor.</u> (Dirac) If G is simple with $n (\geq 3)$ vtcs and if
$$\deg(v) \geq \frac{n}{2} \text{ for each } v \in V(G)$$
then
G is Hamiltonian.

<u>Pf.</u> $\deg(u)+\deg(w) \geq n$ for all $u, w \in V(G)$, satisfying Ore's condition. $\square$.

7.4

Grötzch