Solutions Attendance Quiz for Lecture 7

NAME: Dr. Z.

1. What does it mean for a graph to be **Hamiltonian**?

Ans. to 1: A graph is Hamiltonian of there is cycle that visits every **vertex** exactly once (except that the beginning and end coincide). In other words If the graph has n vertices and we label the vertices, as usual, $1, 2 \dots n$, then the graph is Hamiltonian if there exists a permutation π of $\{1, \dots, n\}$ such that

$$\pi_1 \to \pi_2 \to \pi_3 \to \dots \to \pi_n \to \pi_1 \quad .$$

More formally, calling as usual, the set of edges E, the n edges

$$\{\{\pi_1, \pi_2\}, \{\pi_2, \pi_3\}, \{\pi_3, \pi_4\}, \dots \{\pi_{n-1}, \pi_n\}, \{\pi_1, \pi_n\}\} \quad ,$$

all belong to E.

2. Prove that if G is a simple graph with at least three vertices, and if

$$deg(v) + deg(w) \geq n \quad ,$$

for each **non-adjacent** vertices v and w, then G is Hamiltonian.

Sol. to 2:

See the proof of Theorem 7.1 on p.36 (p. 45 of .pdf) of

`https://sites.math.rutgers.edu/~zeilberg/akherim/wilsongraph.pdf` .

The only statement that is not obvious (and that the author should have elaborated on) is

"It follows that there must be some vertex v_i adjacent to v_1 with the property that v_{i-1} is adjacent to v_n (see Fig. 7.5)

Suppose not and suppose that there a vertices adjacent to v_1. Of course one of them is v_2. Let a be the degree of v_1

Let the set of vertices adjacent to v_1 (except the obvious v_2) be

$$v_{j_1}, v_{j_2}, \dots, v_{j_{a-1}} \quad .$$

If the statement is wrong then we have that v_n is **not** adjacent to the following vertices

$$v_{j_1-1}, v_{j_2-1}, \dots, v_{j_{a-1}-1} \quad .$$

So out of the $n-1$ vertices that are not v_n, $a-1$ of them are not adjacent to v_n and of course neither is v_1 (by assumption). Hence $deg(v_n) \leq (n-1)-1-(a-1) = n-a-1$. Since we assumed that $deg(v_1) = a$ we have

$$deg(v_1) + deg(v_n) \leq a + (n-a-1) = n-1$$

Contradiction!

Hence indeed you can find a v_i as in Fig. 7.5.