Solutions to Attendance Quiz for Lecture 4

NAME: (print!) Dr. Z.

1. Using graph theory solve this classical puzzle

Three missionaries and three cannibals must cross a river using a boat which can carry at most two people, under the constraint that, for both banks, if there are missionaries present on the bank, they cannot be outnumbered by cannibals (if they were, the cannibals would eat the missionaries). The boat cannot cross the river by itself with no people on board.

As follows: Construct a **directed** graph with vertices and edges, where a vertex has labels $[m, c]$, with $0 \le m, c \le 3$ where m is the number of missionaries in the originating bank, and c is the number of cannibals in the originating bank (hence their numbers in the terminal bank are $3 - m$ and $3 - c$. Of course the number of cannibals on either bank should not exceed the number of missionaries, for obvious reasons.

The edges are

$$[m,c] \rightarrow [m-2,c] \quad , \quad [m,c] \rightarrow [m,c-2] \quad , \quad [m,c] \rightarrow [m-1,c-1] \quad , [m,c] \rightarrow [m,c-1] \quad , \quad [m,c] \rightarrow [m-1,c]$$

corresponding to all the boat rides from the originating bank to the terminal bank.

(a) List all the vertices?

Sol. to (a):

$$\{[3,3],[3,2],[3,1],[2,2],[3,0],[0,3],[1,1],[0,2],[0,1],[0,0]\}$$

Explanation: $[2,3],[1,3],[1,2]$ are not good, since the cannibals outnumber the missionaries on the starting river bank.

$[1,0]$,$[2,0]$,$[2,1]$ are not good since the cannibals outnumber the missionaries on the terminal river bank. the remaining pairs are safe.

(b) List all the (directed) edges (for examle $[2,2] \rightarrow [1,1]$.

Sol. of (b):

- Out of $[3,3]$: $[3,3] \rightarrow [3,2] \quad , \quad [3,3] \rightarrow [3,1] \quad , \quad [3,3] \rightarrow [2,2]$
- Out of $[3,2]$: $[3,2] \rightarrow [3,1] \quad , \quad [3,2] \rightarrow [3,0] \quad , \quad [3,2] \rightarrow [2,2]$
- Out of $[3,1]$: $[3,1] \rightarrow [1,1] \quad , \quad [3,1] \rightarrow [3,0]$
- Out of $[3,0]$: Nothing

- Out of $[2,2]$: $[2,2] \to [1,1] \quad , \quad [2,2] \to [0,2]$
- Out of $[1,1]$: $[1,1] \to [0,1] \quad , \quad [1,1] \to [0,0]$
- Out of $[0,3]$: $[0,3] \to [0,2] \quad , \quad [0,3] \to [0,1]$
- Out of $[0,2]$: $[0,2] \to [0,1] \quad , \quad [0,2] \to [0,0]$
- Out of $[0,1]$: $[0,1] \to [0,0]$

(c) See `https://sites.math.rutgers.edu/~zeilberg/Graph25/MCgraph.pdf` .

(d) Solve the puzzle by finding an **alternating path**: Starting with vertex $[3,3]$ and ending with vertex $[0,0]$.

There are four solutions, here is one of them. Can you find the three other ones?

$$[3,3] \to [2,2] \leftarrow [3,2] \to [3,0] \leftarrow [3,1] \to [1,1] \leftarrow [2,2] \to [0,2] \leftarrow [0,3] \to [0,1] \leftarrow [0,2] \to [0,0]$$

Translation into English:

You start with 3 missionaries and 3 cannibals on the left bank (and hence 0 missionaries and 0 cannibals on the right bank)

First Trip Forward: put 1 missionary and 1 cannibal in the boat and once they arrive at the right back we have

2 missionaries and 2 cannibals on the left bank (and hence 1 missionaries and 1 cannibals on the right bank)

First Trip Back: one missionary rows back, now we have

3 missionaries and 2 cannibals on the left bank (and hence 0 missionaries and 1 cannibals on the right bank)

Second Trip Forward: put 2 cannibals in the boat, now we have

3 missionaries and 0 cannibals on the left bank (and hence 0 missionaries and 3 cannibals on the right bank)

Second Trip Back: one cannibal rows back, now we have

3 missionaries and 1 cannibals on the left bank (and hence 0 missionaries and 2 cannibals on the right bank)

Third Trip Forward: put 2 missionaries in the boat, now we have

1 missionaries and 1 cannibals on the left bank (and hence 2 missionaries and 2 cannibals on the right bank)

Third Trip Back: put one missionary and one cannibal in the boat, now we have:

2 missionaries and 2 cannibals on the left bank (and hence 1 missionaries and 1 cannibals on the right bank)

Fourth Trip Forward: put 2 missionaries in the boat, now we have

0 missionaries and 2 cannibals on the left bank (and hence 3 missionaries and 1 cannibals on the right bank)

Fourth Trip Back: put one cannibal in the boat, now we have

0 missionaries and 3 cannibals on the left bank (and hence 3 missionaries and 0 cannibals on the right bank)

Fifth Trip Forward: put 2 cannibals in the boat, now we have

0 missionaries and 1 cannibals on the left bank (and hence 3 missionaries and 2 cannibals on the right bank)

Fifth Trip Back: put one cannibal in the boat, now we have

0 missionaries and 2 cannibals on the left bank (and hence 3 missionaries and 1 cannibals on the right bank)

Sixth (and FINAL) trip Forward:

put two cannibals in the boat, now we have

0 missionaries and 0 cannibals on the left bank (and hence 3 missionaries and 3 cannibals on the right bank)