Solutions to Attendance Quiz for Lecture 18

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1. State and prove Ramsey's theorem about 2-coloring the edges of the complete graph.

Ramsey's Theorem:

Let r and s be any two positive integers. There exists a least positive integer $R(r,s)$, for which every blue-red edge colouring of the complete graph on $R(r,s)$ vertices contains a blue clique on r vertices or a red clique on s vertices.

($R(r,s)$ signifies an integer that depends on both r and s.)

Proof:

Induction on $r+s$. It is clear from the definition that for all n, $R(n,2)=R(2,n)=n$.

This starts the induction. We prove that $R(r,s)$ exists by finding an explicit bound for it. By the inductive hypothesis $R(r-1,s)$ and $R(r,s-1)$ exist.

Lemma:

$$R(r,s) \leq R(r-1,s) + R(r,s-1) \quad .$$

. **Proof** Consider a complete graph on $R(r-1,s)+R(r,s-1)$ vertices whose edges are coloured with two colours. Pick a vertex v from the graph, and partition the remaining vertices into two sets M and N, such that for every vertex w, w is in M if the edge $\{v,w\}$ is blue, and w is in N if the edge $\{v,w\}$ is red. Because the graph has $R(r-1,s)+R(r,s-1)=|M|+|N|+1$ vertices, it follows that either

$|M| \geq R(r-1,s)$

or

$|N| \geq R(r,s-1)$

In the former case, if M has a red K_s then so does the original graph and we are finished. Otherwise M has a blue K_{r1} and so $M \cup \{v\}$ has a blue K_r by the definition of M. The latter case is analogous. QED.