Solutions to Attendance Quiz for Lecture 17

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1. What are the chromatic polynomials (in the variable k) of (i) K_n (ii) C_n (iii) W_n

Comment: 1(ii) and 1(iii) were assigned by mistake. I was going to go over it in class, but didn't have a chance. I didn't expect you to do it from scratch. Since they are instructive I post the solutions here.

Solution to 1(i)

$$P_{K_n}(k) = k(k-1)...(k-n+1) \quad .$$

Because you have k choices of colors for the first vertex, then $k-1$ for the second, then $k-2$ for the third etc (since all vertices must have distinct colors, since this is the complete graph).

Solution to 1(ii) : This is a hard one. Let's do C_3, C_4, C_5 and look for a pattern.

If e is any edge of C_n then $C_n - e$ is P_n and C_n/e is C_{n-1}. So we have the recurrence

$$P_{C_n}(k) = k(k-1)^{n-1} - P_{C_{n-1}}(k) \quad .$$

C_3 is K_3 so

$$P_{C_3}(k) = k(k-1)(k-2) \quad .$$

Using the recurrence

$$P_{C_4}(k) = k(k-1)^3 - k(k-1)(k-2) \quad .$$

$$P_{C_5}(k) = k(k-1)^4 - (k(k-1)^3 - k(k-1)(k-2)) = k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2) \quad .$$

$$P_{C_6}(k) = k(k-1)^5 - (k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2)) = k(k-1)^5 - k(k-1)^4 + k(k-1)^3 - k(k-1)(k-2)$$

So a pattern emerges

$$P_{C_n}(k) = k(k-1)^{n-1} - k(k-1)^{n-2} + \ldots + (-1)^n k(k-1)^3 + (-1)^{n+1} k(k-1)(k-2) \quad .$$

By summing the *geometric series* this simplifies to

Ans. to 1(ii): $P_{C_n}(k) = (k-1)^n + (-1)^n (k-1)$.

Another way: Go to Wolfram Alpha look up the answer and verify that this expression satisfies the recurrence: $P_{C_n}(k) = k(k-1)^{n-1} - P_{C_{n-1}}$ and then check that for $n = 3$ it is equal to $P_{K_3}(k) = k(k-1)(k-2)$.

Solution to 1(iii) : You can color the center dot with k different colors. Once you do that there are $k-1$ colors left to color the perimeter, so once you know $P_{C_n}(k)$, $P_{W_n}(k)$ is easy. Recall that a wheel W_n has a center vertex connected by spokes to $n-1$ vertices on the perimeter that form a cycle. There are k ways to color the center vertex, leaving only $k-1$ colors for the circumference, that is C_{n-1}.

Hence

$$P_{W_n}(k) = kP_{C_{n-1}}(k-1) = k((k-2)^{n-1} + (-1)^{n-1}(k-2)) \quad .$$

2. Prove that the number of ways to color a simple graph G with k colors is always a **polynomial** in k, that of course depends on G.

SOl of 2: By the **deletion-contraction** relation

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k) \quad ,$$

the chromatic polynomial of a graph with m edges is a difference of the chromatic polynomials of graphs with fewer edges.

If a graph has 0 edges (the base case) the chromatic function is k^n, a polynomial!

Assuming that if G has $\leq m-1$ edges the chromatic function is a polynomial, and since a polynomial minus a polynomial is a polynomial, it follows by induction that the chromatic function of a graph with m edges is a polynomial for all m.