NAME: (print!) HOLEN YEE

E-Mail address: hy433@scarletmail.rutgers.edu

MATH 428 (2), Dr. Z. , Exam 2, Wed., Nov. 26, 2025, 10:20-11:40pm, TILLETT-251

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM

No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.

Do not write below this line

---

1. 10 (out of 10)

2. 10 (out of 10)

3. 20 (out of 20)

4. 20 (out of 20)

5. 20 (out of 20)

6. 10 (out of 10)

7. 10 (out of 10)

---

tot.: (out of 100)

100

1. (10 points altogether)
(a) (7 points) Using Euler's formula and the fact that every face must have at least three edges, prove that for a simple planar graph with m edges and n vertices we have

$$m \leq 3n - 6 \quad .$$

Since every face must have at least 3 edges and each edge contributes to two faces, we have $3f \leq 2m$, where f denotes the number of faces. This is the same as $f \leq \frac{2}{3}m$. Combined with Euler's formula, $n-m+f=2$, we get $n-m+\frac{2}{3}m \geq 2$, which is equivalent to $m \leq 3n-6$.

(b) (3 points) Using this fact, prove that K_5 is non-planar.

Suppose for contradiction that K_5 is planar. Then, $m \leq 3n-6$. K_5 has 5 vertices and 10 edges. So $10 \leq 3(5)-6 = 9$. $10 \leq 9$ is a contradiction, so K_5 must be non-planar.

2. (10 points altogether)
(a) (7 points) Using Euler's formula and the fact that every face must have at least three edges, prove that for a simple planar graph that has no triangular faces with m edges and n vertices, we have

$$m \leq 2n - 4 \quad .$$

If a graph has no triangular faces, then every face has at least 4 edges. Since each edge contributes to two faces, we have $4f \leq 2m$, where f denotes the number of faces. This is the same as $f \leq \frac{1}{2}m$. Combined with Euler's formula, $n-m+f=2$, we get $n-m+\frac{1}{2}m \geq 2$, which is equivalent to $m \leq 2n-4$.

(b) (3 points) Using this fact, prove that $K_{3,3}$ is non-planar.

Suppose for contradiction that $K_{3,3}$ is planar. $K_{3,3}$ is bipartite, so it has no triangular faces. Then, $m \leq 2n-4$. $K_{3,3}$ has 6 vertices and 9 edges, so $9 \leq 2(6)-4 = 8$. $9 \leq 8$ is a contradiction so $K_{3,3}$ must be non-planar.

3. (20 points) State (3 points) and prove (17 points) the Five-Color Theorem for Planar graphs.

Statement of Five-Color Theorem: Let G be a simple planar graph. G is 5-colorable.

Proof: We will prove this by induction on the number of vertices, n.

The theorem is trivial for $n \le 5$. Assume that the theorem is true for all simple planar graphs with less than n vertices.

Since G is planar, there exists a vertex, which we'll call v, which has degree at most 5. There are two cases: $\deg(v) \le 4$ and $\deg(v) = 5$.

In the case $\deg(v) \le 4$:

Remove v from G to get the subgraph G'. Since G' has less than n vertices, it is 5-colorable. Add v back to G'. Since v has at most 4 neighbors, there must be a spare color that is not used by its neighbors. Color v with that spare color.

In the case that $\deg(v) = 5$:

Call v's neighbors v_1, v_2, v_3, v_4, v_5. There must be neighbors of v that are not adjacent to each other, since otherwise, $v_1, \dots, v_5$ would form a K_5, making G non-planar. Without loss of generality, say these nonadjacent neighbors are v_1 and v_2. Contract the edges $\{v, v_1\}$ and $\{v, v_2\}$ to form the subgraph G'. Since G' has less than n vertices, it is 5-colorable. Undo the contraction of edges to get back G. Color v_1 and v_2 with the color assigned to the combined vertex vv_1v_2. Since v_1 and v_2 share a color, there must be a spare color not used by v's neighbors. Color v with that spare color.

4. (20 points altogether)
(a) (3 points) State Euler's formula relating the number of vertices, edges, and faces of a planar graph.

Euler's formula: $n - m + f = 2$, where n, m, f denote the number of vertices, edges, and faces respectively.

(b) (17 points) Prove it.

We will prove this by induction on the number of edges. Assume that Euler's formula is true for planar graphs with less than m edges.

If G is a tree:

Then $m = n-1$ and $f = 1$. So $n - m + f = n - (n-1) + 1 = 2$.

If G is not a tree:

Then G must have a cycle. Let e be an edge of a cycle in G. Remove e from G to get the subgraph G', with $n' = n$ vertices, $m' = m-1$ edges, and $f' = f-1$ faces. Since G' has less than m edges, $n' - m' + f' = 2$.

This means $2 = n - (m-1) + f - 1 = n - m + f$.

5. (20 points altogether) (a) (5 points) define the **chromatic function** of a simple graph G, $P_G(k)$.

The chromatic function $P_G(k)$ is a function denoting how many ways you can color G with k colors.

(b) (5 points) *State* the **deletion-contraction** recurrence relation for $P_G(k)$.

$P_G(k) = P_{G-e}(k) - P_{G\setminus e}(k)$, where e is some edge of G.

(c) (7 points) *Prove* the **deletion-contraction** recurrence relation

It is equivalent to show that $P_{G-e}(k) = P_G(k) + P_{G\setminus e}(k)$.

Let e be $\{v, w\}$ for some vertices v, w.

$P_G(k)$ counts all colorings of G-e where v and w are different colors.

$P_{G\setminus e}(k)$ counts all colorings of G-e where v and w are the same color.

So, $P_{G-e}(k) = P_G(k) + P_{G\setminus e}(k)$.

(d) (3 points) Use the **deletion-contraction** recurrence relation to prove that $P_G(k)$ is always a polynomial in k.

If we keep deleting and contracting edges, eventually we get the null graph.

$P_{N_r}(k) = r^k$, which is a polynomial. The sum of polynomials is a polynomial,

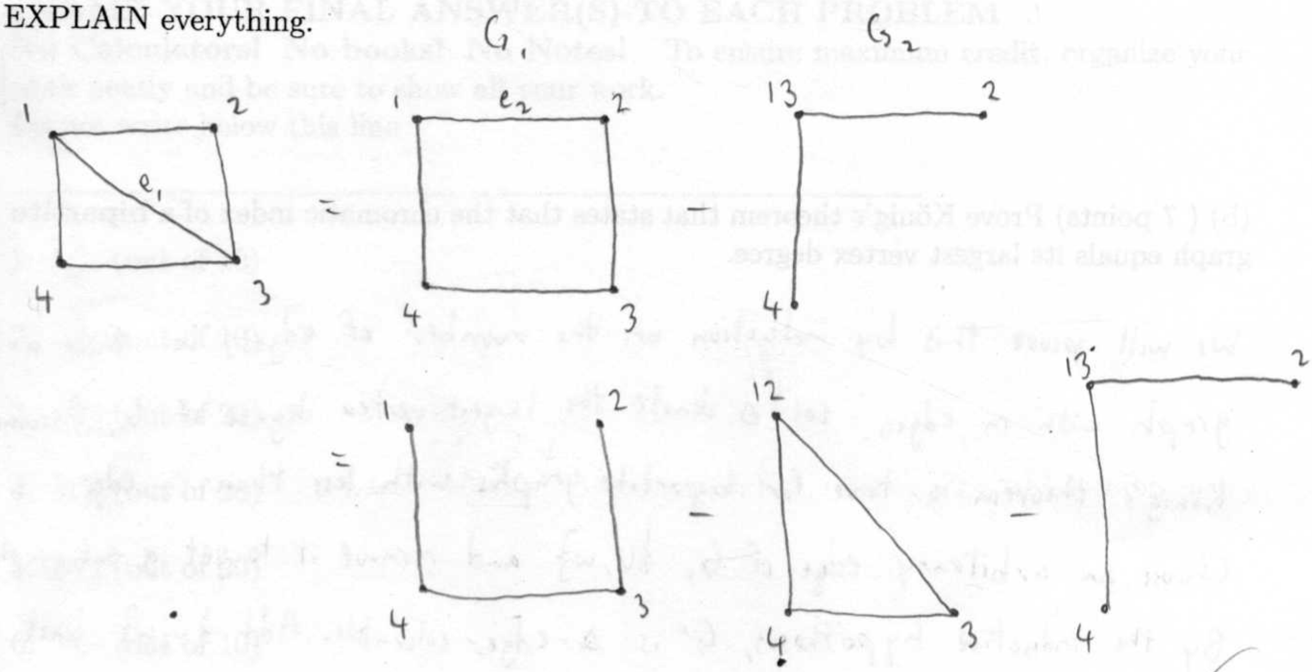
so $P_G(k)$ is a polynomial.

6. (10 points) In how many ways can you color the vertices of the graph G below with 10 colors?

G is the graph with 4 vertices labeled $\{1,2,3,4\}$ and the set of edges is

$$\{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,3\}\}$$

EXPLAIN everything.



$$P_G(k) = k(k-1)^3 - k(k-1)(k-2) - k(k-1)^2$$

$$P_G(10) = 10(9)^3 - 10(9)(8) - 10(9)^2$$

$$= 7290 - 720 - 810$$

$$= \boxed{5760}$$

We use the deletion-contraction recurrence relation first on G with e_1, then on G_1 with e_2. We end up with two trees and one complete graph. For a tree T on n vertices, $P_T(k) = k(k-1)^{n-1}$. For a complete graph on n vertices, $P_{K_n}(k) = k(k-1)\dots(k-n+1)$. Using these formulas and substituting k for 10, we get that there are 5760 ways to color G with 10 colors.

7. **(10 points) (a) (3 points) Define the *chromatic index* of a simple graph.**

The chromatic index is the lowest integer k such that the graph is k-edge-colorable.

(b) (7 points) Prove König's theorem that states that the chromatic index of a bipartite graph equals its largest vertex degree.

We will prove this by induction on the number of edges. Let G be a bipartite graph with m edges. Let Δ denote the largest vertex degree of G. Assume that König's theorem is true for bipartite graphs with less than m edges.
Choose an arbitrary edge of G, {v, w} and remove it to get a subgraph G'.
By the inductive hypothesis, G' is Δ-edge-colorable. Add {v, w} back to G.
v and w are each not incident to at least one color of edge out of the Δ colors.
If v and w are missing the same color, color {v, w} with that color.
If v and w are missing different colors:
Say that v is missing α and w is missing β. Let H denote the subgraph of G which can be reached by an α-β path from v. W is not in H, since G is bipartite and has even cycles. If w is in H, then w would be missing α.
Reverse all α- and β-colored edges in H. Now v is missing β and w is missing β. Color {v, w} β.