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MATH 428 (2), Dr. Z. , Exam 2, Wed., Nov. 26, 2025, 10:20-11:40pm, TILLETT-251

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM

No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.

Do not write below this line

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1. 10 (out of 10)
2. 10 (out of 10)
3. 20 (out of 20)
4. 20 (out of 20)
5. 20 (out of 20)
6. 10 (out of 10)
7. 10 (out of 10)

tot.: (out of 100)

100

1. (10 points altogether)

(a) (7 points) Using Euler's formula and the fact that every face must have at least three edges, prove that for a simple planar graph with m edges and n vertices we have

$$m \leq 3n - 6 \quad .$$

→ Euler's states that $n - m + f = 2$

① → For a simple planar graph, all faces have at least 3 edges:

$$m \geq \frac{3f}{2}$$

$$\frac{2m}{3} \geq f$$

② Plugging this into Euler's:

$$n - m + \frac{2m}{3} \geq 2$$

$$3n - 3m + 2m \geq 6$$

$$3n - m \geq 6$$

$$\boxed{m \leq 3n - 6}$$

Using algebra, we yield

(b) (3 points) Using this fact, prove that K_5 is non-planar.

① For K_5, we have:

$n = 5$

$m = \frac{5(4)}{2} = 10$

② Plugging this into $m \leq 3n - 6$:

$10 \leq 15 - 6 \Rightarrow 10 \leq 9$

K_5 does not satisfy $m \leq 3n-6$. Since $m \leq 3n - 6$ is only satisfied with planar graphs, K_5 is NON-PLANAR.

2. (10 points altogether)

(a) (7 points) Using Euler's formula and the fact that every face must have at least three edges, prove that for a simple planar graph that has no triangular faces with m edges and n vertices, we have

$$m \leq 2n - 4 \quad .$$

① For a simple planar graph with no triangles, all faces have at least 4 edges:

$$m \geq \frac{4f}{2}$$

$$m \geq 2f$$

$$\frac{m}{2} \geq f$$

② Plugging this into Euler's

$$n - m + \frac{m}{2} \geq 2$$

$$2n - 2m + m \geq 4$$

$$2n - m \geq 4$$

$$\boxed{m \leq 2n - 4}$$

Using algebra we yield

(b) (3 points) Using this fact, prove that $K_{3,3}$ is non-planar.

① For $K_{3,3}$: (It is bipartite, so no triangles exist):

$n = 6$

$m = 3 \cdot 3 = 9$

② Plugging into $m \leq 2n - 4$:

$9 \leq 2(6) - 4$

$9 \leq 8 \rightarrow$ $K_{3,3}$ does not satisfy $m \leq 2n - 4$. Therefore, $K_{3,3}$ is NON-planar.

3. (20 points) State (3 points) and prove (17 points) the Five-Color Theorem for Planar graphs.

→ The 5-color theorem states that all simple planar graphs can be colored with just 5 colors.

→ Proof:

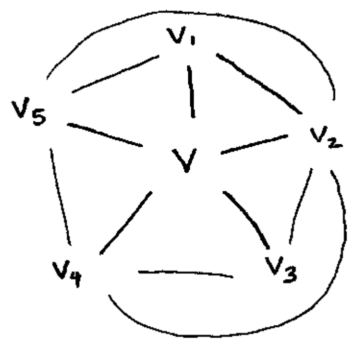
In a planar graph, there must be at least one vertex whose degree ≤ 5. (Else, $m \geq \frac{6n}{2} \rightarrow m \geq 3n$, which contradicts $m \leq 3n-6$)

Taking a vertex v with degree ≤ 5, there are 2 cases:

$\boxed{\text{Case 1}}$ $\deg(v) \leq 4$

- We can remove v and color the graph with 5 colors. Restore v, and color it with the remaining color.

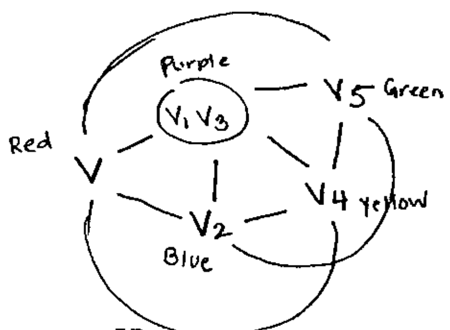
$\boxed{\text{CASE 2}}$ $\deg(v) = 5$



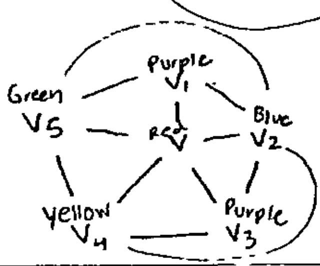
① In the graph, there MUST be at least 2 vertices that do not share an edge (Otherwise, it would be K_5, which is non-planar). For this case, let us use V_1 and V_3

② Contract V_1 and V_3 such that they become 1 vertex with the previously shared neighbors.

③ Color the new graph with 5 colors



④ Restore V_1 and V_3, color them with the same color used for V_1V_3. Therefore, the graph is 5-colorable.



4. (20 points altogether)
(a) (3 points) State Euler's formula relating the number of vertices, edges, and faces of a planar graph.

For a planar graph where f = # of faces, m = # of edges, and n = # of vertices, then

$$n - m + f = 2$$

(b) (17 points) Prove it.

We prove by induction:

① Base case: $n=1$, $m=0$:

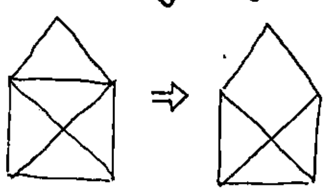
$f=1$

"– the infinity" face

$n-m+f = 1-0+1 = 2 = 2$

The base case satisfies the theorem

② Taking any graph G, remove an edge:



The graph now has:

$n \to n' = n$

$m \to m' = m-1$

$f \to f' = f-1$

We claim the new graph, G', also satisfies Euler's:

$n' - m' + f' = 2$

Subbing back our original values

$(n) - (m-1) + (f-1) = 2$

$n - m + 1 + f - 1 = 2$

$n - m + f = 2$

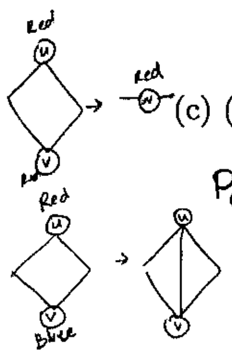
Thus, by the inductive hypothesis, for all planar graphs: $n-m+f=2$

5. (20 points altogether) (a) (5 points) define the **chromatic function** of a simple graph G, $P_G(k)$.

The chromatic function, $P_G(k)$, is a function in K that states how many ways a simple graph G is k-colorable given k colors.

(b) (5 points) *State* the **deletion-contraction** recurrence relation for $P_G(k)$.

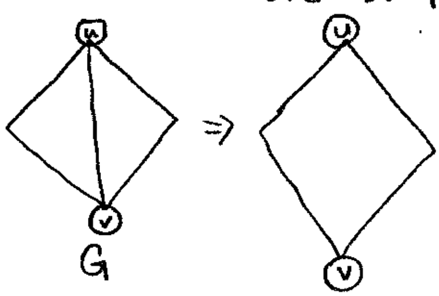
$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$



(c) (7 points) *Prove* the **deletion-contraction** recurrence relation

$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$ can be rewritten as $P_{G-e}(k) = P_G(k) + P_{G/e}(k)$. Looking at the graph of $G-e$, there are 2 possible cases:

where $\{u,v\}$ is the deleted edge,



CASE 1: u and v are different colors, Restoring the edge would give us the same as $P_G(k)$

CASE 2: u and v are the same color, we contract u and v, giving us $P_{G/e}(k)$

Therefore $P_{G-e}(k) = P_G(k) + P_{G/e}(k)$, and

$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$

(d) (3 points) Use the **deletion-contraction** recurrence relation to prove that $P_G(k)$ is always a polynomial in k.

We know that $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$

Using base cases $P_{N_n}(k) = K^n$ and $P_{K_n}(k) = K(K-1)\cdots(K-n+1)$ for the null and complete graph

$P_{G-e}(k)$ and $P_{G/e}(k)$ have to also be polynomials by induction.

Since the two parts $P_{G-e}(k)$ and $P_{G/e}(k)$ are polynomials, then $P_G(k)$ is also a polynomial in k.

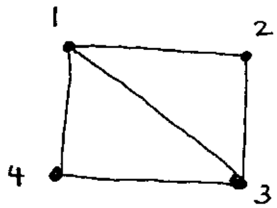
6. (10 points) In how many ways can you color the vertices of the graph G below with 10 colors?

G is the graph with 4 vertices labeled $\{1,2,3,4\}$ and the set of edges is

$P_G(10) = ?$

$$\{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,3\}\}$$

EXPLAIN everything.

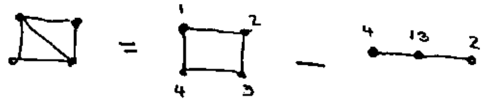


→ We are trying to find $P_G(10)$ for G.

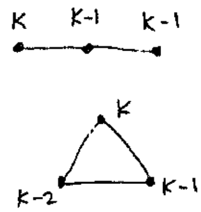
→ Let us first find $P_G(k)$ and plug in $k=10$ into the resulting chromatic polynomial

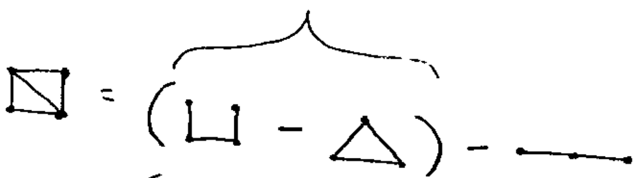
① Use the deletion-contraction recurrence relation

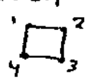
$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$:



deletion / contraction of $\{1,3\}$





deletion / contraction of $\{1,2\}$ in 

= This is a tree, $P_T(k) = k(k-1)^{n-1}$, so

$k(k-1)^3$

This is a complete graph, K_3, $P_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1)$, so

$k(k-1)(k-2)$

This is a tree, $P_T(k) = k(k-1)^{n-1}$, so

$k(k-1)^2$

② The Chromatic Polynomial is =

$$P_G(k) = k(k-1)^3 - k(k-1)(k-2) - k(k-1)^2$$

③ Plugging in $k=10$:

$$P_G(10) = 10(9)^3 - 10(9)(8) - 10(9)^2$$

$$= 7290 - 720 - 810$$

$$= 5760$$

↳ There are 5760 ways to color G with 10 colors.

5760
+ 720
6480
+ 810
7290

81
× 9
729

7290
− 720
6570
− 810
5760

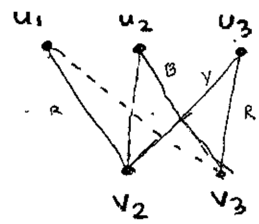
7. (10 points) (a) (3 points) Define the *chromatic index* of a simple graph.

The Chromatic index of a simple graph, $\chi'(G)$, is the smallest K such that G is K-EDGE colorable.

(b) (7 points) Prove König's theorem that states that the chromatic index of a **bipartite** graph equals its largest vertex degree.

For a bipartite graph, $\chi'(G) = \Delta$, where Δ is the largest vertex degree.

① Given a bipartite graph, REMOVE an edge



② Color the remaining edges with Δ colors. In doing so,

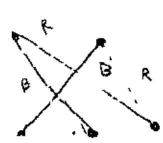
u is missing at least 1 color $(\Delta-1)$

v is missing at least 1 color $(\Delta-1)$

③ 2 cases:

1) If u and v are missing the same color, restore the edge and color it the missing color. We are done.

2) If u and v are missing different colors, exchange the colors of the edges and restore the removed edge with a remaining color. We are done.



In doing so, the bipartite graph is Δ-edge-colorable.