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MATH 428 (2), Dr. Z. , Exam 1, Thurs. Oct. 21, 2025, 10:20-11:40pm, TILLETT-251

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM

No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.

Do not write below this line

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1. 10 (out of 10)

2. 10 (out of 10)

3. 20 (out of 20)

4. 20 (out of 20)

5. 20 (out of 20)

6. 10 (out of 10)

7. 10 (out of 10)

Excellent!

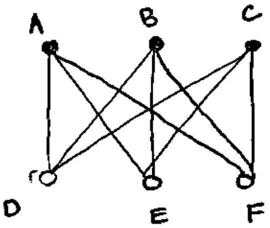
tot.: (out of 100)

100

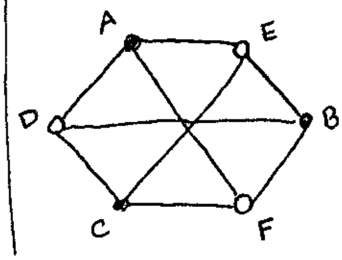
1. (10 points) Prove that $K_{3,3}$ is non-planar.

A non-planar graph is one that cannot be drawn as a plane graph (aka will always have "fake" vertices when drawn.

① First let us draw $K_{3,3}$:

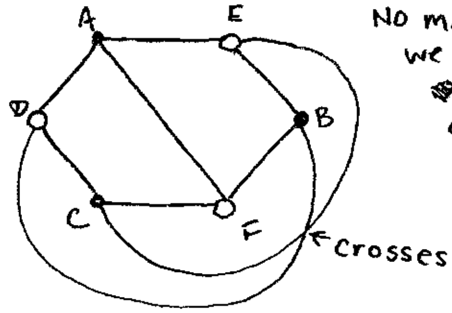


② Keeping the cycle, let's redraw $K_{3,3}$:



We can seperate edges "inside" and "outside" the cycle. Here, clearly there are 3 "inside" edges that cross. We

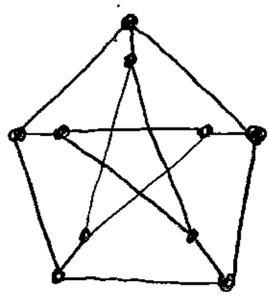
③ We can only keep 1 edge "inside". For this ex. we keep AF.



No matter how we draw the remaining edges, BD an CE will alway intersect, making this a non plane Graph.

Repeating this for all combinations of "inside" edges yields the same result. Thus, $K_{3,3}$ is non-planar

2. (10 points altogether) (a) (5 points) Draw the Petersen graph



(b) (5 points) Find a Eulerian cycle, or explain why it does not exist.

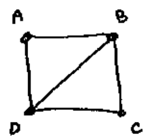
$\boxed{\text{NO EULERIAN CYCLE EXISTS.}}$

For a graph to be eulerian (has a eulerian cycle), all the vertices must have even degree. The Petersen graph is a Regular graph whose vertices all have degree 3. Since 3 is not an even number, it is non Eulerian, and thus does not contain a Eulerian cycle.

3. (20 points altogether) (a) (3 points) Define what it means for a simple graph to be *Hamiltonian*

For a simple graph to be Hamiltonian, it must have a Hamiltonian Cycle, such that each vertex is traversed exactly once. (exception to the first and last vertex being the same in a cycle).

ex.



(b) (10 points) State, but do not prove, Ore's theorem about a sufficient condition for a simple graph to be Hamiltonian

For any simple Graph G where $n \geq 3$, it is Hamiltonian if for any two non-adjacent vertices u, and V:

$$\boxed{deg(u) + deg(V) \geq n}$$

(c) (7 points) State Dirac't theorem about a sufficient condition for a simple graph to be Hamiltonian, and show how it follows from Ore's theorem.

Dirac's: For any simple graph G where $n \geq 3$, it is Hamitonian if for any vertex V:

$$\boxed{deg(V) \geq \frac{n}{2}}$$

→ If this is true, then for another vertex u: $deg(u) \geq \frac{n}{2}$

Adding these two together yields Ore's:

$$deg(V) + deg(u) \geq \frac{n}{2} + \frac{n}{2}$$

$$= \boxed{deg(V) + deg(u) \geq n}$$

4. (20 points altogethr)

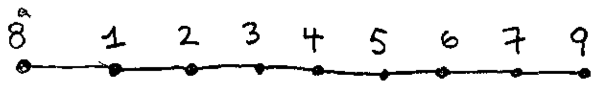
(a) (10 points) Draw the labeled tree whose Prüfer Code is

1234567

A: $(\cancel{1},\cancel{2},\cancel{3},\cancel{4},\cancel{5},\cancel{6},\cancel{7})$

C: $(\cancel{1},\cancel{2},\cancel{3},\cancel{4},\cancel{5},\cancel{6},7,\cancel{8},9)$

$\{8,1\}, \{1,2\}, \{2,3\}$ $\{3,4\}$ $\{4,5\}$ $\{5,6\}$ $\{6,7\}$ $\{7,9\}$



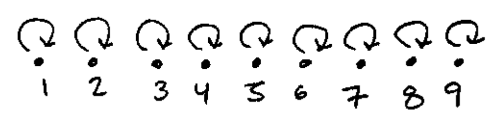
(b) (10 points) Draw the doubly rooted labeled tree whose Joyal Code is

123456789

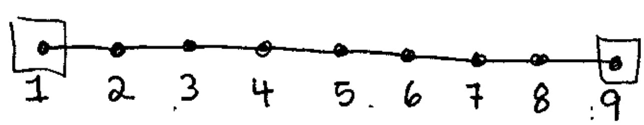
Indicate the **primary root** and the **secondary root**

maps to 1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7 8 9



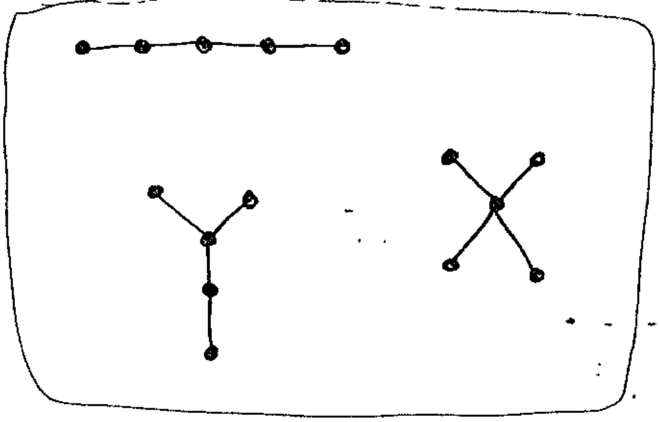
$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,2\,3\,4\,5\,6\,7\,8\,9 \end{pmatrix}$ 2 line form



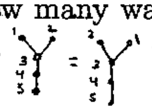
primary root = 1

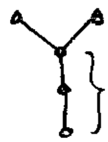
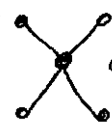
Secondary root = 9

5. (20 points altogether) ~~(a) (10 points)~~ Draw all the *unlabeled trees* with 5 vertices



(b) (10 points) For each of them, state how many ways can you label them. What is the total number?

(ie )

Unlabeled tree	 $\frac{n!}{2}$ ways n! ways to choose, but since it is symmetrical (ie 12345 = 54321) then we divide by 2:	 only the identity of these 3 pts matter.		← only the identity of the point in the center matters, and there are 5 choices
# of ways to label	$\frac{n!}{2}$ ways $= \frac{5!}{2} = \frac{5\cdot4\cdot3\cdot\cancel{2}\cdot1}{\cancel{2}} = 60$ $\boxed{60}$	$5\cdot4\cdot3 = 60$ ways $\boxed{60}$	5 ways $\boxed{5}$	

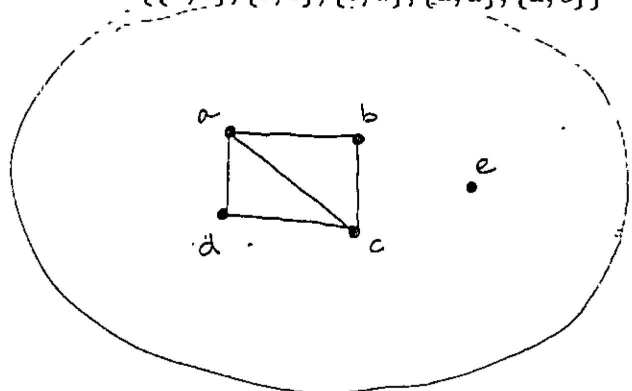
$60 + 60 + 5 = \boxed{125 \text{ ways}}$

Supported by Cayle's: $n^{n-2} \Rightarrow 5^{3-2} \Rightarrow 5^3 = 125$ ✓

9

6. (10 points altogether) (a) (2 points) Draw the graph whose set of vertices is $\{a, b, c, d, e\}$ and whose set of edges is

$$\{\{a,b\},\{b,c\},\{c,d\},\{a,d\},\{a,c\}\}\ .$$

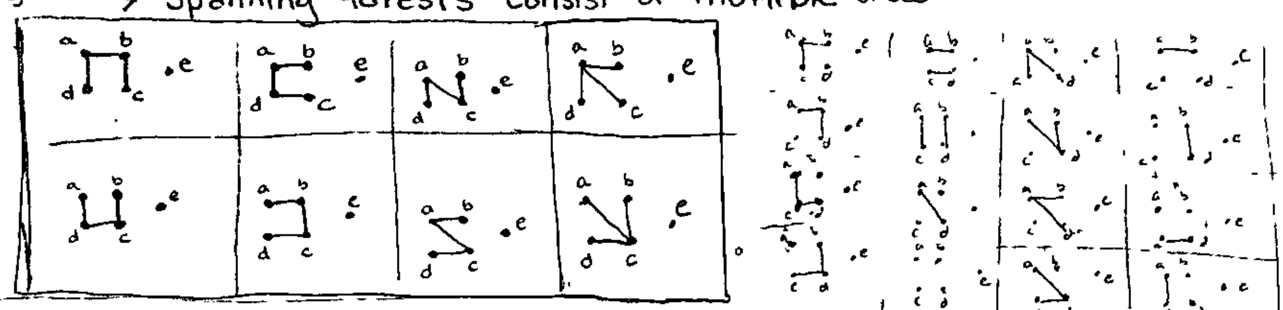


(b) (8 points) Draw all its spanning trees or forests.

→ It is impossible to produce any spanning trees for this graph, as it is disconnected. Since this graph has 5 vertices, a spanning tree has n-1, or 4 edges. Any combination of 4 edges with this graph would produce a cycle, which violates the criterion of a tree.

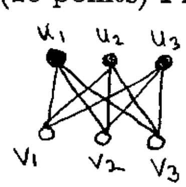
Keeping {a,b,c,d} and {e} as a tree, we yield these outcomes

→ Spanning forests consist of multiple trees



If we were to find all possible combinations of trees, by breaking up {a,b,c,d}, this produces many more outcomes

7. (10 points) Prove that if G is a bipartite graph, then every cycle has even length.



In a bipartite graph, there are two sets of vertices, we can call them U (where $\{u_1, \dots u_{n_1}\}$) and V {where $\{v_1, \dots v_{n_2}\}$ such that there is not edge connecting any vertices of U together or any vertices of V together. Thus, any path of odd length would either go from $U \to V$ or $V \to U$, ending in a different "set" than originated.

Conversely, ~~the~~ any walk of even length would return to the original set. Since cycles start and end at the same vertex (and thus the same ~~starting~~ set U or V), it is thus required for any cycle to have even length.