Real Quiz 9

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1. Prove that every simple planar graph is 5-colorable.

Proof by Induction: Base Case: $n=1$ • $n=2$ •—• Both graphs are 5-colorable ✓

Inductive Step: Suppose a simple planar graph with $n-2, n-1$ vertices is 5-colorable. Let G be a simple planar graph with n vertices. Since G is planar, G must have a vertex v such that $\deg(v) \leq 5$.
If $\deg(v) < 5$, remove v and its incident edges. Then $G \setminus v$ is a simple planar graph with $n-1$ vertices and by the inductive hypothesis $G \setminus v$ can be colored with 5 colors. Add v and its incident edges back. Since v has at most 4 adjacent vertices there remains at least 1 color not used among those vertices adjacent to v. Color v with such a color.

If $\deg(v)=5$, then there exist $v_1, v_2, \ldots, v_5$ such that each v_i is adjacent to v. If each v_i is adjacent to every other v_j, then G contains K_5 and cannot be planar. This is a contradiction, so there must exist some v_i and v_j that are not adjacent. Contract v and v_i and v and v_j. Then the graph created has $n-2$ vertices and by the inductive hypothesis can be colored with 5 colors. Reversing the contractions, color v_i and v_j with the color assigned to v in the contracted graph. Now the vertices adjacent to v are at most 4 different colors, so there remains at least 1 color not used among the vertices adjacent to v. Color v with such a color.

Therefore, by induction, if G is a simple planar graph then G is 5-colorable. ∎

Real Quiz 9

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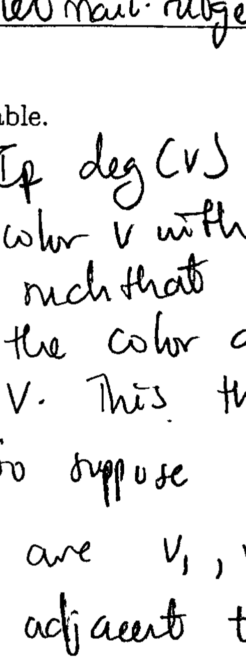
1. Prove that every simple planar graph is 5-colorable.

Solution: Proof by induction on the number of vertices. The result is trivial for simple planar graphs with fewer than six vertices. So suppose G is a simple planar graph with n vertices and all simple planar graphs with n-1 vertices are 5-colorable. Based on Theorem 13.6, we know that a simple planar graph contains a vertex with degree of at most 5. Therefore, G contains a vertex v that has a degree of at most 5. If we delete v and its incident edges, we obtain a graph G' with n-1 vertices, and is thus 5-colorable. Our aim now is to assign v with one of the 5 colors, by obtaining a 5-coloring of G.

If deg(v) < 5, then we can simply color v with one of the avaliable colors such that color of v is different from the color of the vertices adjacent to v. This then completes the proof.

So suppose deg(v) = 5. Then there are $v_1, v_2 --, v_5$ (vi) vertices adjacent to v.



If the vertices (vi) are mutually adjacent, then the graph G contains the non-polar K_5 as a subgraph, which is impossible. So there must be two vertices of vi (say v_1 & v_3) that are not adjacent. If we contract the two edges vv_1 and vv_3, we then get a graph with fewer than n vertices

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This implies that this graph is 5-colorable. We then reinstate the two edges v_1 and v_3 by coloring them with the original color assigned to v. Therefore, A 5-coloring of G is obtained by coloring v with a color different from the vertices (v_i) adjacent to v.

∴ This proves that every simple Planar Graph is 5-colorable