Solutions to Attendance Quiz for Lecture 9

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1. Prove that a tree with n vertices has at least two leaves (aka *endpoints*)

Sol. to 1: We first prove that there is at lesat one endpoint (i.e. vertex of degree 1). Suppose not, then whenever you walk randomly you can keep going without ever retracing. Since the graph is finite, sooner of later you would wind up in a previously visited vertex, thereby forming a cycle, contradicting that it is a tree (a tree, by definition, is a graph without cycles).

Let the degrees be $d_1, d_2, \ldots, d_n$, where $d_1 \leq d_2 \leq \ldots \leq d_n$. We already know that $d_1 = 1$. We claim that d_2 is also 1. Suppose that $d_2 \geq 2$, then also $d_i \geq 2$, for all $2 \leq i \leq n$.

Then

$$d_2 + d_3 + \ldots + d_n >= 2n - 2$$

Since the number of edges of the tree is $n - 1$, it follows from the *handsahking lemma* that

$$d_1 + d_2 + \ldots + d_n = 2(n - 1)$$

Since $d_1 = 1$, this implies

$$d_2 + \ldots + d_n = 2n - 3$$

So we get that the assumption that there is only one endpoint the contradiction that $2n - 3$ is larger than $2n - 2$, nonsense. QED.