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MATH 428 (2), Dr. Z. , Exam 2, ~~Monday~~ Thursday, Nov. ~~27~~ 30, 2023, 12:10-1:20pm, TILLET-105

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM

No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.

Do not write below this line

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1. 10 (out of 10)

2. 10 (out of 10)

3. 20 (out of 20)

4. 20 (out of 20)

5. 20 (out of 20)

6. 10 (out of 10)

7. 10 (out of 10)

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tot.: (out of 100) EXCELLENT!

100

10

1. (10 points altogether)
(a) (7 points) Use Euler's formula and the fact that each face is bounded by at least three edges, to find an upper bound for the number of edges that a simple planar graph with n vertices can have.

Solution: Proof: For a simple planar graph with n vertices, each face is bounded by at least 3 edges. And we know that every edge bounds 2 faces. ∴ We get the inequality; $3f \leq 2m$ where f = no of faces, and m = no of edges. Recall, Euler's formula; $n - m + f = 2$ ∴ $f = 2 + m - n$

Substitute into the inequality

$$3(2+m-n) \leq 2m$$
$$6+3m-3n \leq 2m$$
$$3m-2m \leq 3n-6$$
$$m \leq 3n-6$$

∴ $m \leq 3n-6$ is the upper bound for the number of edges that a simple planar graph with n vertices can have.

(b) (3 points) Using this fact, prove that K_5 is non-planar.

Solution: Upper bound; $m \leq 3n-6$.

For K_5, $n = 5$ vertices; $m = \frac{n(n-1)}{2} = \frac{5(4)}{2} = \frac{20}{2} = 10$ edges.

∴ $10 \leq 3n-6 = 3(5)-6 = 15-6 = 9$

∴ $10 \leq 9$ which is wrong

∴ This proves that K_5 is non-planar

10

2. (10 points altogether)
(a) (7 points) Use Euler's formula and the fact that each face is bounded by at least three edges, to find an upper bound for the number of edges that a simple planar graph that has **no triangular faces**, with n vertices can have.

Solution: If a simple planar graph has no triangular faces, then each face is bounded by at least 4 edges. And we know that every edge bounds 2 faces ∴ We get the inequality $4f \le 2m$ where f = no of faces, m = no of edges. Recall Euler's formula: $n - m + f = 2$

$\therefore f = 2 + m - n$; $4f \le 2m$

$4(2+m-n) \le 2m$

$8 + 4m - 4n \le 2m$

$4m - 2m \le 4n - 8$

$2m \le 4n - 8$

divide both sides by 2

$\therefore m \le 2n - 4$

∴ The upper bound for the no of edges that a simple planar graph that has no triangular faces with n vertices is $m \le 2n - 4$

(b) (3 points) Using this fact, prove that $K_{3,3}$ is non-planar.

Solution: Since every cycle in $K_{3,3}$ is of even length, then it contains no triangular faces. ∴ We use this inequality $m \le 2n - 4$

$n = 3 + 3 = 6$ vertices

$m = 3(3) = 9$ edges

$\therefore 9 \le 2(6) - 4 = 12 - 4 = 8$

$\therefore 9 \le 8$ which is wrong

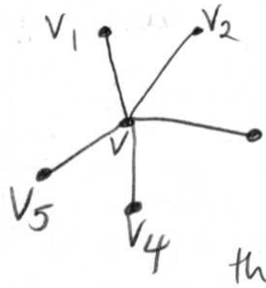
∴ This proves that $K_{3,3}$ is non-planar

√ 20

3. (20 points) Prove the Five-Color Theorem for Planar graphs

Solution: Proof by induction on the number of vertices. The result is trivial for simple planar graphs with fewer than six vertices. Suppose G is a simple planar graph with n vertices, and all simple planar graphs with $n-1$ vertices are 5-colourable. Based on Theorem 13.6, we know that a simple planar graph must contain a vertex with degree of at most 5. So let v be a vertex in G with a degree of at most 5. If we delete v and its incident edges, the graph that remains has $n-1$ vertices, and is thus 5-colourable. Our aim now is to color v with one of the 5 colours, by obtaining a 5-colouring of G.

If $\deg(v) < 5$, we can simply color v with one of the available colors such that the color of v is different from the vertices adjacent to v. This completes the proof. So suppose $\deg(v) = 5$. Therefore, there are 5 vertices ($v_i = v_1, v_2 \dots, v_5$) that are adjacent to v



If these vertices are mutually adjacent, then the graph contains non-planar K_5 as a subgraph which is impossible. So the vertices (v_i) must have at least two vertices (say v_1 & v_3) that are non adjacent. If we contract the two edges vv_1 & vv_3, we obtain a graph with fewer than n vertices, and is thus 5-colourable. We then reinstate the vertices v_1 & v_3 by coloring them with the color originally assigned to v. A 5-colouring of G is then obtained by colouring v with a colour different from the vertices (v_i) adjacent to v.

This proves the 5-color theorem for planar graphs.

20

4. (20 points altogether)
(a) (3 points) State Euler's formula relating the number of vertices, edges, and faces of a planar graph.

Solution: Euler's Formular : For a connected planar graph with v vertices, e edges and f faces, then $v - e + f = 2$

(b) (17 points) Prove it.

Solution: Proof by induction : i) For $e = 0$, •, $v = 1$, $f = 1$

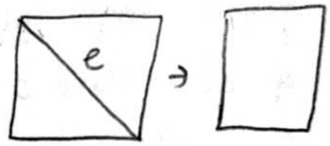
$\therefore v - e + f = 1 - 0 + 1 = 1 + 1 = 2$, which holds true

ii) For $e = 1$, •—•, $v = 2$, $f = 1$ $\therefore v - e + f = 2 - 1 + 1 = 2$, which holds true

iii) For $e = 1$, O, $v = 1$, $f = 2$ $\therefore v - e + f = 1 - 1 + 2 = 0 + 2 = 2$, which holds true

If we have a bigger Graph G. Let G' be the graph obtained after removing an edge e from G. i.e $G' = G - e$

Case 1: $G' = G - e$ and the graph is still connected

G: 

G'

$\therefore v' = v$, $e' = e - 1$, $f' = f - 1$

$\therefore v' - e' + f' = v - e + \cancel{1} + f - \cancel{1} = v - e + f$ and based on induction this is equals to 2. $\therefore$ It holds true.

Case 2: $G' = G - e$ but the graph becomes disconnected. This forms 2 components G_1 and G_2 $\therefore G_1' = (v_1', e_1', f_1')$

$G_2' = (v_2', e_2', f_2')$

$\therefore v = v_1' + v_2'$, $e = e_1' + e_2' + 1$, $f = f_1' + f_2' - 1$

$\therefore v - e + f = v_1' + v_2' - e_1' - e_2' - 1 + f_1' + f_2' - 1$

$= (v_1' - e_1' + f_1') + (v_2' - e_2' + f_2') - 1 - 1$

Based on Induction $= 2 + \cancel{2} - \cancel{2} = 2$, which holds true.

$\therefore$ All of these cases prove Euler's Formula of a planar graph which is $v - e + f = 2$

√ 20

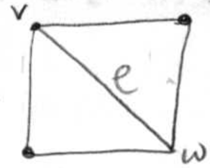
5. (20 points altogether) (a) (5 points) define the **chromatic function** of a simple graph G, $P_G(k)$.

Solution: The chromatic function of a simple graph G, $P_G(k)$, is the number of ways to color the vertices of G using k colors, such that no two adjacent vertices have the same color.

(b) (5 points) (5 points) *State* the **deletion-contraction** recurrence relation for $P_G(k)$

Solution: Statement: If G is a simple graph, and $G-e$ and $G \backslash e$ are graphs obtained from G by deleting and contracting an edge e. Then

$$P_G(k) = P_{G-e}(k) - P_{G\backslash e}(k)$$

G: 

(c) (7 points) *Prove* the **deletion-contraction** recurrence relation

Proof: Let edge $e = vw$. The number of k-colorings of $G-e$ in which v and w have different colors remains unchanged if the edge e is drawn joining v & w. Therefore, this equates to $P_G(k)$. Similarly, the number of k-colorings of $G-e$ in which v and w have the same color remains unchanged if v and w are identified. Therefore, this equates to $P_{G\backslash e}(k)$. Therefore, the total number $P_{G-e}(k)$ of k-colorings of $G-e$ is $P_G(k) + P_{G\backslash e}(k)$

$\therefore$ This implies that $P_G(k) = P_{G-e}(k) - P_{G\backslash e}(k)$

(d) (3 points) Use the **deletion-contraction** recurrence relation to prove that $P_G(k)$ is always a polynomial in k.

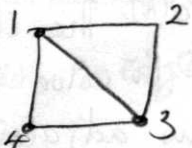
Proof: We can continue the procedure above by choosing edges in $G-e$ and $G\backslash e$ by deleting and contracting them. The process terminates when no edges are left in the remaining graphs. That is, when all the remaining graphs become null graphs. We know that the chromatic function of a null graph is a polynomial, which is k^r (where r is the number of vertices). It follows by recursion of the application that the chromatic function of G must be a sum of polynomials, and must itself thus be a a polynomial in K

$$\begin{array}{r} 7200 \\ -1440 \\ \hline 5760 \end{array} \qquad \begin{array}{r} 16 \\ +8 \\ \hline 72 \end{array} \qquad \begin{array}{r} 720 \\ +720 \\ \hline 1440 \end{array} \qquad 720 \times 10 = 7200$$

✓ (10)

6. (20 points) In how many ways can you color the vertices of the graph G below with 10 colors?

G is the graph with 4 vertices labeled $\{1, 2, 3, 4\}$ and the set of edges is $\{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,3\}\}$

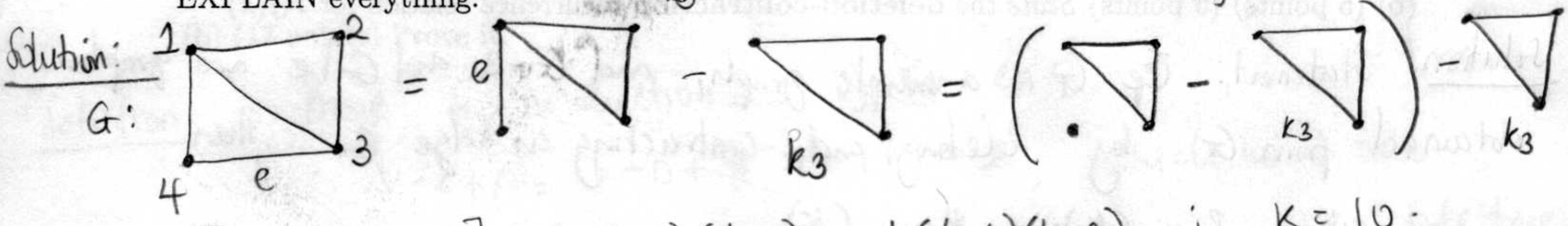


$k = 10$, Recall; $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$

Recall; $P_{K_3}(k) = k(k-1)(k-3+1) = k(k-1)(k-2)$

EXPLAIN everything.

Solution:



$$\therefore P_G(k) = k[k(k-1)(k-2)] - k(k-1)(k-2) - k(k-1)(k-2) \quad ; \quad k = 10.$$

$$\therefore P_G(10) = 10\,[10(9)(8)] - 10(9)(8) - 10(9)(8) = 7200 - 1440 = 5760 \text{ ways}$$

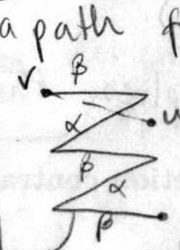
7. (10 points) (a) (3 points) Define the *chromatic index* of a simple graph.

Solution: Chromatic index of a simple graph, $X'(G)$, is the smallest number of colors needed to color the edges of G such that no adjacent edges have the same color.

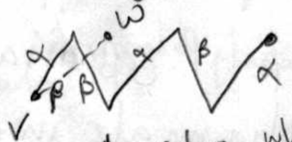
✓ (10)

(b) (7 points) Prove König's theorem that states that the chromatic index of a **bipartite** graph equals its largest vertex degree.

Solution. Let the largest vertex degree be Δ. Let G be a bipartite graph, with all its edges but one being colored. Suppose such edge is edge vw. This means that at least one color is missing in vertex v and at least one color is missing in vertex w. If for some case that the same color is missing in both v and w, the color edge vw with that color. If that is not the case, then let α be the color missing in v and let β be the color missing in w, and let $H_{\alpha\beta}$ be a subgraph of G that consists of vertex v and those edges and vertices in G that can be reached from vertex v by a path from vertex v.



Since G is a bipartite graph, then $H_{\alpha\beta}$ can not consist of vertex v. So we can interchange the colors α and β in the subgraph without affecting w and the rest of the coloring. Therefore, color edge vw with β, this then completes the coloring of the edges of G.



$\therefore$ This implies that the edges of G can be colored with at most Δ-colors, which is the largest vertex degree $\therefore X'(G) = \Delta$.