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MATH 428 (2), Dr. Z. , Exam 2, ~~Monday, Nov. 27,~~ Thurs, Nov 30 2023, 12:10-1:20pm, TILLET-105

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM

No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.

Do not write below this line

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1. 10 (out of 10)

2. 10 (out of 10)

3. 20 (out of 20)

4. 20 (out of 20)

5. 20 (out of 20)

6. 10 (out of 10)

7. 10 (out of 10)

tot.: (out of 100)

100

Excellent

1. (10 points altogether) $V-E+F=2$ $3f \le 2m$

(a) (7 points) Use Euler's formula and the fact that each face is bounded by at least three edges, to find an upper bound for the number of edges that a simple planar graph with n vertices can have.

$m \le 3n-6$

Proof: Since every face is bounded by at least 3 edges, $3f \le 2m$.
By Euler's formula, $n-m+f=2$, so $f=2+m-n$.
Then $3f=3(2+m-n)=6+3m-3n \le 2m$ and $6-3n \le -m$, so $m \le 3n-6$. □

✓

(b) (3 points) Using this fact, prove that K_5 is non-planar.

Proof by Contradiction: Suppose K_5 is planar. Then, by part (a), since K_5 has 5 vertices and 10 edges, $10 \le 3(5)-6=9$.
This is a contradiction as $10>9$, so K_5 cannot be planar. □

✓

2. (10 points altogether) $V-E+F=2$

(a) (7 points) Use Euler's formula and the fact that each face is bounded by at least three edges, to find an upper bound for the number of edges that a simple planar graph that has **no triangular faces**, with n vertices can have. $3f \le 2m$

$4f \le 2m$

$m \le 2n-4$

Proof: Since the graph has no triangles, every face must have at least 4 edges and $4f \le 2m$.

By Euler's formula, $n-m+f=2$, so $f=2+m-n$.

Then $4f=4(2+m-n)=8+4m-4n \le 2m$ and $8-4n \le -2m$, so $m \le 2n-4$ □

(b) (3 points) Using this fact, prove that $K_{3,3}$ is non-planar.

Proof by contradiction: Suppose $K_{3,3}$ is planar. Then by part (a) since $K_{3,3}$ has 6 vertices and 9 edges, $9 \le 2(6)-4=8$.

This is a contradiction, as $9>8$, so $K_{3,3}$ cannot be planar. □

10

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3. (20 points) Prove the Five-Color Theorem for Planar graphs

Proof by Induction (on the number of vertices)

Base Case: $n=1$  $n=2$  Both graphs are five-colorable.

Inductive Step: Suppose a planar graph with $n-1$ or $n-2$ vertices is five colorable.

Let G be a planar graph with n vertices. Since G is planar, G has a vertex v with degree at most 5.

Case 1: Suppose $\deg(v) \leq 4$. Remove v from G to create G', a planar graph with $n-1$ vertices. By the inductive hypothesis, G' can be colored with 5 colors. Add back vertex v. Since v is adjacent to at most 4 vertices, there are at most 4 colors used among the vertices adjacent to v. Thus, there remains at least one color with which v can be legally colored and G is 5-colorable.

Case 2: Suppose $\deg(v)=5$. Then there exist $v_1, v_2, \ldots, v_5$ such that v_i is adjacent to v for $1 \leq i \leq 5$. If each v_i is adjacent to all v_j for $j \in \{1, \ldots, 5\} \setminus \{i\}$, then G contains K_5 and is nonplanar. This is a contradiction, so there must exist some v_i and v_j not adjacent to each other, say v_1 and v_3. Then we can contract vv_1 and vv_3 to get G'', a planar graph with $n-2$ vertices. By the inductive hypothesis, we can color G'' with 5 colors. Reversing the contraction, we can color v_1 and v_3 the color of v in G'' and then there are at most 4 colors used among the vertices adjacent to v. Thus, there is at least 1 color with which v can be legally colored and G is 5-colorable.

By induction, a planar Graph G with n vertices is five colorable for all n. □

4. (20 points altogether)

(a) (3 points) State Euler's formula relating the number of vertices, edges, and faces of a planar graph.

For a connected planar graph with n vertices, m edges, and f faces, $n-m+f=2$.

(b) (17 points) Prove it.

Proof by Induction on m: Base Case: $m=0$ • $n=1, f=1$ $1-0+1=2$ ✓

$m=1$ •—• $n=2, f=1$ $2-1+1=2$ ✓ The formula holds for $m=0, m=1$

Inductive Step: Suppose for a connected planar graph with $m-1 \geq 1$ edges, $n-m+f=2$. Let G be a connected planar graph with m edges.

Case 1: Suppose G is a tree. Then since G has m edges, G has $m+1$ vertices and since G is a tree G has one face. ✓
Thus $n=m+1, f=1$, and $n-m+f=m+1-m+1=2$.

Case 2: Suppose G is not a tree. Then G contains at least one cycle. Remove an edge from this cycle to create G', a connected planar graph with $m'=m-1$ edges, $n'=n$ vertices, and $f'=f-1$ faces.
By the inductive hypothesis, $n'-m'+f'=2$, so $n-m+f = n'-m'-1+f'+1 = n'-m'+f'=2$.

By induction, for a connected planar graph with n vertices, m edges, and f faces, $n-m+f=2$. □

✓

20

5. (20 points altogether) (a) (5 points) define the **chromatic function** of a simple graph G, $P_G(k)$.

The chromatic function $P_G(K)$ is the number of ways that the simple graph G can be legally colored with K colors.

(b) (5 points) (5 points) *State* the **deletion-contraction** recurrence relation for $P_G(k)$

$$P_G(K) = P_{G-e}(K) - P_{G/e}(K)$$

where e is an edge in G.

20

(c) (7 points) *Prove* the **deletion-contraction** recurrence relation

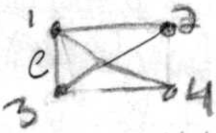
Proof: Remove any edge e from G to create $G-e$, where $e=\{v,w\}$. The number of ways to color $G-e$ with K colors such that v and w are different colors is equivalent to $P_G(K)$, as adding back edge e will not affect the legality of such a coloring.
The number of ways to color $G-e$ with K colors such that v and w are the same color is equivalent to $P_{G/e}(K)$, as contracting e will not affect the legality of the coloring when v and w are the same color.
Thus, $P_{G-e}(K) = P_G(K) + P_{G/e}(K)$ and $P_G(K) = P_{G-e}(K) - P_{G/e}(K)$ □

(d) (3 points) Use the **deletion-contraction** recurrence relation to prove that $P_G(k)$ is always a polynomial in k.

Proof: Case 1: G is a null graph with r vertices. Then $P_G(K) = K^r$, which is a polynomial in K

Case 2: G is not a null graph. By the deletion-contraction recurrence, we can find $P_G(K)$ by deleting and contracting edges. Repeat this process until we are left with no edges. Then $P_G(K)$ is equal to the sums and differences of the chromatic functions of null graphs, which are polynomials by Case 1. Thus, $P_G(K)$ is a polynomial in K □

6. (20 points) In how many ways can you color the vertices of the graph G below with 10 colors?
G is the graph with 4 vertices labeled $\{1, 2, 3, 4\}$ and the set of edges is



$$\{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,3\}\}$$

EXPLAIN everything. Use $P_G(K) = P_{G-e}(K) - P_{G/e}(K)$. Path of length 4, K_3

Path of length 3

$P_G(K) = \ldots = K(K-1)^3 - (K(K-1)(K-2)) - K(K-1)^2 = K(K-1)[(K-1)^2 - (K-2) - (K-1)]$ $P_G(10) = 10(9)[9^2 - (8) - (9)] = (90)[81-17] = (90)(64) = \boxed{5760 \text{ ways}}$ ✓

81
−17
64

90
×64
360
5400
5760

20

7. (10 points) (a) (3 points) Define the *chromatic index* χ' of a simple graph.

The chromatic index of a simple graph G is the smallest number of colors with which the edges of G can be legally colored, called $\chi'(G)$.

(b) (7 points) Prove König's theorem that states that the chromatic index of a **bipartite** graph equals its largest vertex degree.

Proof by Induction on m: Base Case: $m=1$ $\Delta = 1$ and $\chi'(G) = 1$.
Inductive Step: Suppose a bipartite graph G' with $m-1 \geq 1$ edges and largest vertex degree Δ' has $\chi'(G') = \Delta'$. Let G be a bipartite graph with m edges and largest vertex degree Δ. Then by Vizing's Theorem, $\Delta \leq \chi'(G) \leq \Delta + 1$. Take any edge e such that $e = \{v, w\}$ and remove it. Then by the inductive hypothesis, the graph can be colored with Δ colors, as the largest vertex degree is less than or equal to Δ. Then vertices v and w are each missing a color among their incident edges.
Case 1: Suppose v and w are each missing the color α. Then add back e and color it α.
Case 2: Suppose v is missing color α and w is missing color β. Take the subgraph of all paths starting at v with edges colored only α or β. Since G is a bipartite graph, w cannot be included in any of these paths as w is incident to an edge of color α, v is incident to an edge of color β, and a path of only α and β colored edges when added to e create an odd length cycle. Therefore, we can switch the colors α and β in the subgraph so that now v and w are both missing β, and we can color e β to create a Δ coloring of G. Thus $\chi'(G) = \Delta$.
By induction, $\chi'(G) = \Delta$ if G is bipartite. □