NAME: (print!) George Basta

E-Mail address: gnb35@scarletmail.rutgers.edu

MATH 428 (2), Dr. Z. , Exam 1, Thurs. Oct. 26, 2023, 12:10-1:30pm, TILLET-105

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM
No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.
Do not write below this line

---

1. 10 (out of 10).

2. 10 (out of 10)

3. 20 (out of 20)

4. 20 (out of 20)

5. 20 (out of 20)

6. 10 (out of 10)

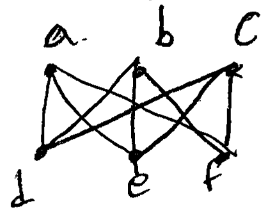
7. 10 (out of 10)

tot.: (out of 100)

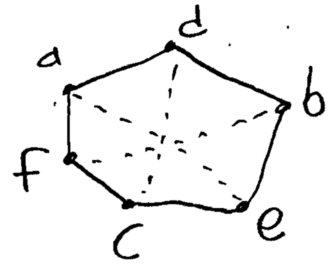
100

EXCELLENT!

1. (10 points) Prove that $K_{3,3}$ is non-planar.

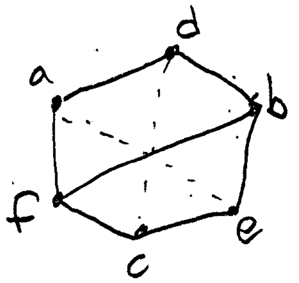


Draw the largest cycle in $K_{3,3}$.

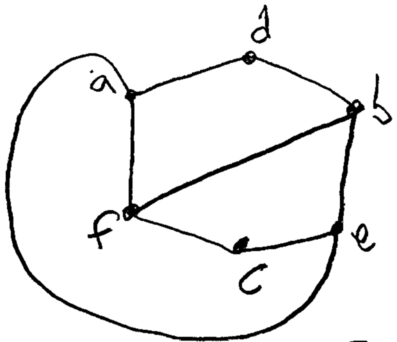


Now WLOG, draw edge $\{b f\}$ inside the cycle

(could also draw outside, argument is similar)



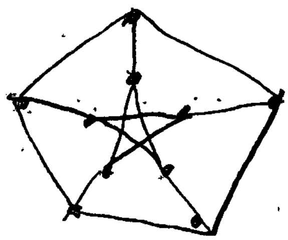
We can't draw edge $\{a e\}$ inside, so draw it outside, s..



Similarly, $\{d c\}$ can't be drawn inside, But now, $\{d c\}$ also can't be drawn outside without intersection,

Thus, $K_{3,3}$ has at least one intersection, so non-planar

2. (10 points altogether) (a) (5 points) Draw the Petersen graph



(b) (5 points) Find a Eulerian cycle, or explain why it does not exist.

A Eularian Cycle does not exist

The Peterson Graph is Regular of degree 3, So all vertices have degree 3.

A graph has a eularian cycle $\Longleftrightarrow$ All degree's are even

Since no degree's are even, the graph does not have a Eularian Cycle

3. (20 points altogether) (a) (3 points) Define what it means for a simple graph to be *Hamiltonian*

A simple graph is Hamiltonian if we can draw a closed path such that every vertex is in the path.

(A path is a walk that doesn't repeat edges or vertices. A closed path is a path that finished where it started)

(b) (10 points) State, but do not prove, Ore's theorem about a sufficient condition for a simple graph to be Hamiltonian

ORE: If a simple graph with $n \geq 3$ vertices is connected and satisfies

$$\deg(v) + \deg(u) \geq n$$

for any two nonadjacent vertices v, u,

then the graph is Hamiltonian

(c) (7 points) State Dirac't theorem about a sufficient condition for a simple graph to be Hamiltonian, and show how it follows from Ore's theorem.

Dirac: If a simple graph with $n \geq 3$ vertices is connected and satisfies

$$\deg(v) \geq \frac{n}{2}$$

for all vertices v, then the graph is Hamiltonian.

Follows from ORE's theorem because clearly, $\frac{n}{2} + \frac{n}{2} \geq n$ for any two nonadjacent vertices, so by ore's theorem

$\deg(v) + \deg(u) \geq n$

So the graph is Hamiltonian

4. (20 points altogethr)
(a) (10 points) Draw the labeled tree whose Prüfer Code is

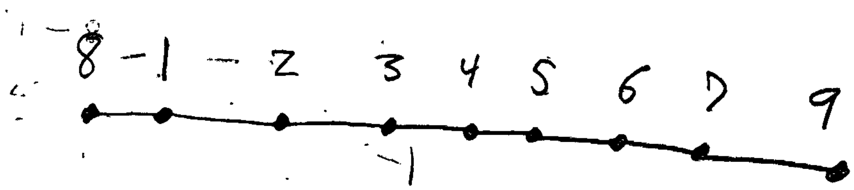
1234567.

Consideration: {1, 2, 3, 4, 5, 6, 7}

7+2 = 9 vertices. {1, 2, 3, 4, 5, 6, 7, 8, 9}

Connect the smallest not under consideration to first in considera

8-1,
1-2
2-3
3-4
4-5
5-6
6-7, final edge is 7-9

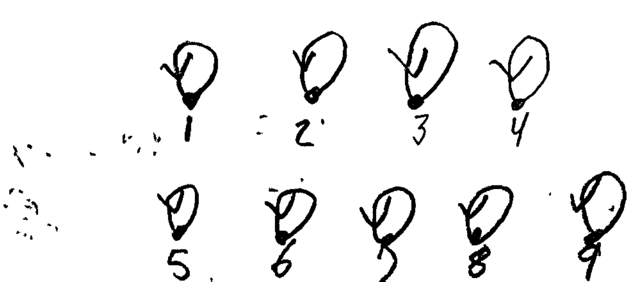


(b) (10 points) Draw the doubly rooted labeled tree whose Joyal Code is

123456789

Indicate the **primary root** and the **secondary root**

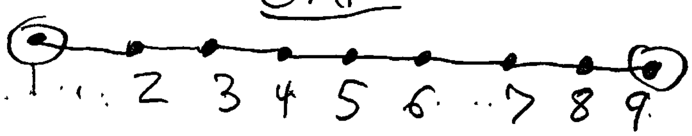
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$



Joyal Mapping

$\binom{1}{1}\binom{2}{2}\binom{3}{3}\binom{4}{4}\binom{5}{5}\binom{6}{6}\binom{7}{7}\binom{8}{8}\binom{9}{9}$

Graph



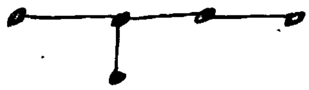
1 primary root

9 Secondary root

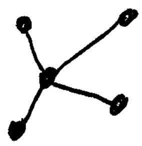
5. (20 points altogether) (a) (10 points) Draw all the *unlabeled trees* with 5 vertices

1

2



3



(b) (10 points) For each of them, state how many ways can you label them. What is the total number?

For 1: $\frac{5!}{2} = 60$ ways (divide by 2 to remove the reverse identical labels)

For 2: $\left(\frac{4!}{2}\right) \cdot 5 = 60$ ways ($\frac{4!}{2}$ ways to label P_4, after having 5 options of labeling the vertex sticking out)

For 3: 5 ways
(5 ways to label center vertex, labels outside don't matter)

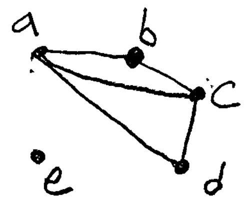
So $60 + 60 + 5 = 125$ total ways to label $n=5$ vertex trees

Cayley's Theorem: $n^{n-2} = 5^{5-2} = 5^3 = 125$ ways to label

So this matches the theoretical value

6. (10 points altotether) (a) (2 points) Draw the graph whose set of vertices is $\{a, b, c, d, e\}$ and whose set of edges is

$$\{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, c\}\}$$



(b) (8 points) Draw all its spanning trees

{ }

No matter what we do, we can't span all the vertices since no edge connects to e, so the amount of spanning trees is the empty set. Spanning Trees only exist for connected graphs.

(by definition, a spanning tree is a tree connecting every vertice)

7. (10 points) Prove that if G is a bipartite graph, then every cycle has even length

$K_{1,n}$: tree has no cycles so statement is vacuously true,

$K_{m,m}$ for $m \geq 2$ has all vertices with at least degree 2, so there must exist a cycle

We know G has a cycle. Pick an arbitrary cycle

$V_1 \rightarrow V_2 \rightarrow \ldots \rightarrow V_i \rightarrow V_1$

& since G is Bipartite, we can split the vertices to be in 2 disjoint sets A & B.

If V_1 is in A, V_2 is in B & so on, forcing V_i in B. Since we go back to V_1 in A to complete the cycle

Note these are pairs of 2, so these arbitrary cycle has a length multiple of 2, so it is even length.