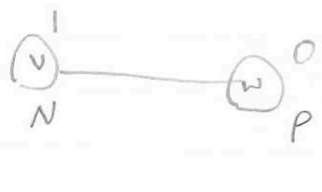
1. $\Rightarrow$: We prove the contrapositive by induction on the # of vertices in the graph of the game. For $n = 2$ we have the following:



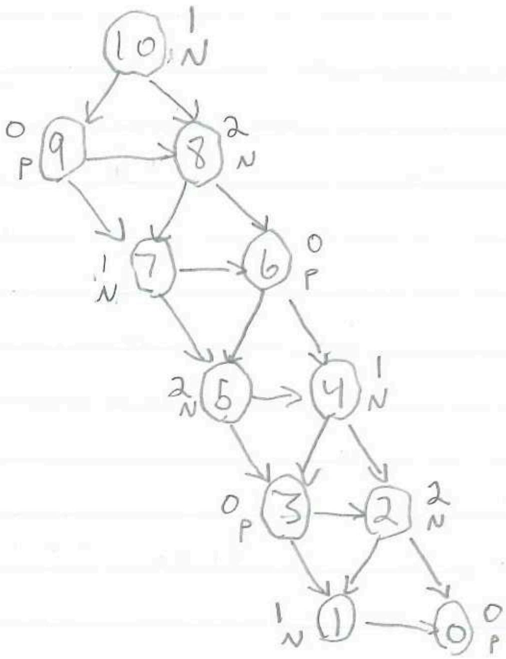
, so the position v is winning as desired.

Consider a game w/ n-vertices/states and supp. $f(v) > 0$. Then $\exists$ vertex $w \in G$ s.t. $f(w) = 0$. Remove v from the graph. Since the game graph is acyclic, the game proceeding from w terminates and is unaffected by the removal of v. Since $f(w) = 0$, all of the vertices adj to w have $f(x) > 0$. By the IH, all of these vertices are winning positions, so w is losing. Adding v back into G, we see that v is winning since it is adj. to the losing position w.

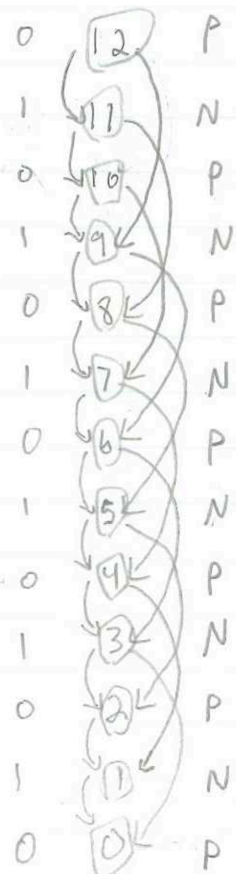
$\Leftarrow$: Suppose $f(v) = 0$. Then every vertex adj. to v, has $f(x) > 0$. By "$\Rightarrow$", these vertices are winning positions, so v is losing.

- agent - causation: A → B → C

2. G_1



3.



4. $G_1 \times G_1 \times G_2$: Initial pos. $v = (8, 6, 5)$

$$F(v) = F_1(8) \oplus F_1(6) \oplus F_2(5)$$

$$\begin{array}{rc} \oplus & 010 \\ & 000 \\ \oplus & 001 \\ \hline & 011 \end{array} = 3_{10} > 0$$

Since $F(v) > 0$, it is an N-position.
A winning move makes $F(w) = 0$. This can be done by removing 1 from the first pile. Then:

$$F(v') = F(7, 6, 5)$$

$$= \begin{array}{rc} \oplus & 0\ 0\ 1 \\ \oplus & 0\ 0\ 0 \\ & 0\ 0\ 1 \\ \hline & 0 \end{array}$$ which is a P-position.

causation: $A \to R \to C$

5. $F(i) = i$. We show this holds by induction.
For $n = 0$, 0 is a sink, so $f(0) = 0$.
Then $F(i) = \text{mex}\{f(i-1), \ldots, f(1), F(0)\}$
$= \text{Mex}\{i-1, \ldots, 1, 0\}$ by IH
$= i$ as desired.

The winning move is to remove all n-counters so that the next player immediatly loses.

6. 4-pile nim: current pos. $v = (4, 5, 6, 8)$

Then
$$f(v) = \begin{array}{r} 0100 \\ \oplus\ 0101 \\ \oplus\ 0110 \\ \oplus\ 1000 \\ \hline 1111 \end{array} = 15_{10} > 0$$

Thus v is an N-position. A winning move is to remove 1 counter from the 8-pile. This leaves the next player in a P-position since

$v' = (4, 5, 6, 7)$
$$\Rightarrow f(v') = \begin{array}{r} 0100 \\ 0101 \\ 0110 \\ 0111 \\ \hline 0000 \end{array} = 0_{10}.$$