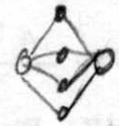
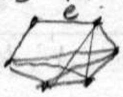
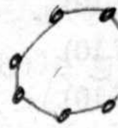
1. (10 points, 2 each) Which of the following graphs are planar? Explain!
(a) $K_{2,4}$ (b) K_3 (c) K_4 (d) K_6 (e) C_8

a) $K_{2,4}$ is planar as it can be drawn as 

b) K_3 is planar as it can be drawn as 

c) K_4 is planar as it can be drawn as 

d) K_6 is nonplanar as it has a subgraph contractable to K_5
subgraph:  Contracte:  K_5

e) C_8 is planar as it can be drawn as 

2. (10 points) If it is your turn to move in a 3-pile Nim, with the the first, second, and third piles having 6, 11, and 15 pennies respectively, what would you do if you want to guarantee to win.

$f(6,11,15) = f(6) \oplus f(11) \oplus f(15) = 6 \oplus 11 \oplus 15 = 0110_2 + 1011_2 + 1111_2 = 0010_2 =$
$= 2$. $\quad 0110_2 + 1011_2 = 1101_2 = 13$

$f(6,11,13) = 0$.

Remove 2 pennies from the third pile to win.

3. (10 points) In the course of using the "keep finding an augmented path until none are left" algorithm to solve the maximum flow problem for a certain network, the following augmented path was found, from the source, vertex 1, to the sink, vertex 9, where the current flow along an edge is indicated in *italics* and its capacity is indicated in **boldface**

$$1 \xrightarrow{\mathit{3},\mathbf{7}} 3 \xrightarrow{\mathit{5},\mathbf{11}} 2 \xleftarrow{\mathit{5},\mathbf{5}} 7 \xleftarrow{\mathit{3},\mathbf{5}} 5 \xrightarrow{\mathit{6},\mathbf{17}} 9 \ .$$

B: 4 +3   6 +3   5 −3   3 −3   11 +3   δ = 3

(a) (2 points) Explain why this is a legal augmenting path. (b) (6 points) Find the new flow along each edge. (c) (2 points) By how much did it increase?

a) This is a legal augmenting path because it starts at the source and ends at the sink, the current flow for all forward edges is greater than or equal to zero and less than the capacity, and the current flow for all backwards edges is greater than zero and less than or equal to the capacity.

b) $1 \xrightarrow[C=7]{\text{Flow}=7} 3 \xrightarrow[C=11]{\text{Flow}=9} 2 \xleftarrow[C=5]{\text{Flow}=2} 7 \xleftarrow[C=5]{\text{Flow}=0} 5 \xrightarrow[C=17]{\text{Flow}=14} 9$

c) The flow increased by 3.

4. (10 points) (a) (2 points) Give an example of a 3×5 Latin rectangle, using the symbols $1, 2, 3, 4, 5$.

(b) (3 points) Calling the columns the 'girls' (Ms. 1, ..., Ms. 5), and the entries the 'boys' (Mr. 1, ..., Mr. 5), formulate, in the *matrinomy* language, the problem of continuing the 3×5 Latin rectangle into a 4×5 Latin rectangle, by adding a new row on top.

(c) (5 points) Find a matching, and use it to creatie a 4×5 Latin rectangle whose bottom three rows are the same as in (a).

a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$

b) Each girl Ms.K has two boys that she knows, Mr. i and Mr. j, where i and j are the entries missing from the Kth column. A complete matching, where each girl is married to one of the boys she knows and no two girls share a husband, will create a new row to add to create a 4x5 Latin rectangle.

c) G1: {4,5}
G2: {1,5}
G3: {1,2}
G4: {2,3}
G5: {3,4}

Boys	Girls
5	1
1	2
2	3
3	4
4	5

$\begin{bmatrix} 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$

Matching: Ms1 + Mr5, Ms2 + Mr1, Ms3 + Mr2, Ms4 + Mr3, Ms5 + Mr4 all deg even   2 deg odd

5. (10 points, 2 each) Which of the following graphs have a Eulerian cycle? Eulerian path? Explain.
(a) K_{10} (b) K_{101} (c) P_{100} (the path with 100 vertices) (d) C_{1001} (the cycle with 1001 vertices) (e) The Petersen graph.

a) K_{10} has neither since it has 10 vertices of degree 9, which is odd
b) K_{101} has an Eulerian cycle as every vertex has degree 100, which is even
c) P_{100} has an Eulerian path as by definition, P has only two vertices of odd degree
d) C_{1001} has an Eulerian cycle as every vertex has degree 2, which is even
e) The Petersen graph has neither since it contains 10 vertices of degree 3, which is odd.

6. (10 points) (a) (5 points) State Hall's theorem about Stable Matchings, in the language of matrimony (with girls and boys some of whom know each other) (b) (5 points) State Hall's theorem about Stable Matchings, in the language of transversals where you are given n subsets of $\{1, \ldots, n\}$, $S_1, \ldots S_n$.

a) For n girls and $m \geq n$ boys, there exists a complete matching if and only if every group of $k \leq n$ girls collectively knows at least k boys.

b) For a family $\mathcal{F} = \{S_1, \ldots, S_n\}$, with $S_i \subseteq \{1, \ldots, n\}$ for $1 \leq i \leq n$, there exists a transversal if and only if for all $K \subseteq \{1, \ldots, n\}$, $|\bigcup_{i \in K} S_i| \geq |K|$

7. (10 points) Prove Hall's theorem. You can use any (correct) proof that you know.

Proof by induction on the number of girls: Base Case: $m=1$. Then this girl knows at least 1 boy, so she can marry him. Thus, there exists a complete matching.

Inductive step: Suppose for $m \geq 1$ girls, if any subgroup of k' girls collectively knows at least k' boys then there exists a complete matching. Suppose you have $m+1$ girls, such that every subgroup of k girls collectively knows at least k boys.

Case 1: Suppose every group of k girls ($k < m+1$) collectively knows at least $k+1$ boys. Then pick any girl and marry her to one of the boys she likes. Then for the remaining m girls, every subgroup of $k \leq m$ girls knows at least k boys collectively, so by the inductive hypothesis there exists a complete matching.

Case 2: there exists a group of $k < m+1$ girls who know exactly k boys collectively. Then since $k \leq m$, by the inductive hypothesis, there exists a complete matching where these k girls marry the k boys. Now for the remaining $m+1-k$ girls suppose there exists a subgroup of h girls who know less than h unmarried boys. Then these h girls and the m married girls collectively know less than $m+h$ boys, which contradicts our assumption. Thus, any subgroup of h of the remaining $m+1-k$ girls must collectively know at least h boys. Since $m+1-k \leq m$, by the inductive hypothesis there exists a complete matching where the remaining girls marry the remaining boys. Thus, there exists a complete matching for this group of $m+1$ girls.

Therefore, by induction, Hall's Theorem is true. $\square$

8. (10 points) A certain simple graph with 101 vertices has the property that it is regular of degree 51 (i.e. every vertex has exactly 51 neighbors). Can you conclude that it has a Hamiltonian cycle? If yes, explain what theorem you are using. If no, also explain why not. $\frac{101}{2}$

This has a Hamiltonian Cycle by Dirac's theorem as $n=101$ and the degree of every vertex is $51 = \frac{102}{2} > \frac{101}{2}$.

$\binom{m+n-2}{m-1} = \binom{6}{3} = \frac{6!}{3!\,3!} = \frac{2\cdot 6\times5\times4\times3\times2}{3\times2\times1} = 20$

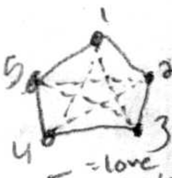
$24/2 = 4$

9. (10 points) In a party of 20 people, any of the 190 pairs of people either love each other or hate each other (they are never indifferent). Are you guaranteed that you can either find 4 people who all love each other (i.e. all the 6 relationships between them are of love) or 4 people who all hate each other (i.e. all the 6 relationships between them are of hate). Explain!

Yes, Since $m=n=4$, $\binom{m+n-2}{m-1} = \binom{6}{3} = \frac{6!}{3!\,3!} = \frac{6\times5\times4}{3\times2} = 2\times5\times2 = 20$

so $R(4,4) \leq 20$ and a party of 20 people is guaranteed to have either 4 people who all hate each other or 4 people who all love each other.

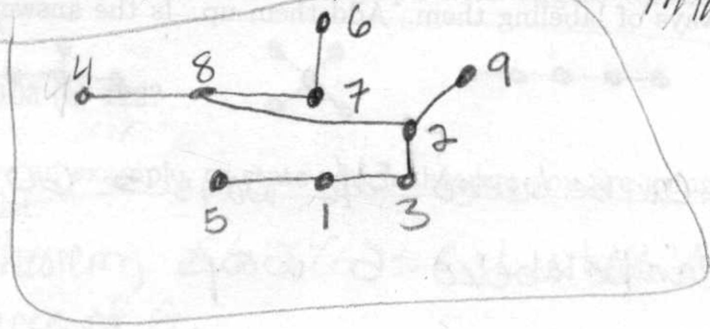
10. (10 points). Construct a party with 5 guests, such that you can **neither** find three people who love each other **nor** three people who hate each other. Calling these people 1, 2, 3, 4, 5, indicate for each of the ten pairs of guests whether they love each other or hate each other.



1 and 2 love each other.
1 and 3 hate each other.
1 and 4 hate each other.
1 and 5 love each other.
2 and 3 love each other.
2 and 4 hate each other.
2 and 5 hate each other.
3 and 4 love each other.
3 and 5 hate each other.
4 and 5 love each other

Consider: {1 2 3 4 5 6 7 8 9}

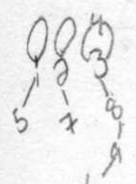
11. (10 points). Find the labeled tree, on 9 vertices, whose Prüfer code is 8132782 . 8132782

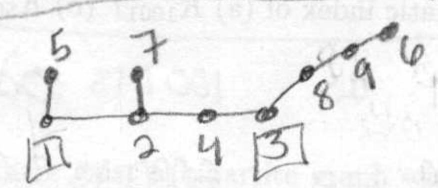


1234
1243

12. (10 points). Find the doubly-rooted labeled tree, on 9 vertices whose Joyal code is 124319238, indicate the primary root and the secondary root.

123456789



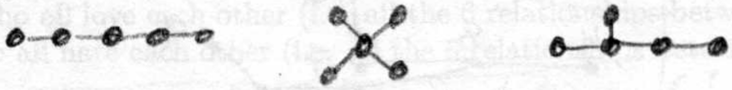


124319238 ✓

Primary Root: 1

Secondary Root: 3

13. (10 points) List all the unlabeled trees with 5 vertices. For each of them describe the number of ways of labeling them. Add them up. Is the answer consistent with Cayley's theorem?



 Can be labeled $5!/2$ ways $= 60$ ways

$\frac{5\times4\times3\times\cancel{2}}{\cancel{2}} = 6$

 Can be labeled 5 ways

 can be labeled $5\times 4!/2$ ways $= 60$ ways

In total there are 125 labeled trees on 5 vertices. This is consistent with Cayley's theorem as $5^{5-2} = 5^3 = 125$.

$\chi(K_n) \begin{cases} n, & n \text{ odd} \\ n-1, & n \text{ even} \end{cases}$

edges!

14. (10 points) What is the chromatic index of (a) K_{1001}? (b) K_{5000}? Explain.

a) $\chi'(K_{1001}) = 1001$ as 1001 is odd

b) $\chi'(K_{5000}) = 4999$ as 5000 is even

15. (10 points, 2 each) Does there exist a simple graph with largest vertex degree 100 whose chromatic index ~~is~~:

χ'

(a) 90 (b) 100 (c) 101 (d) 103 (e) 122?

X √K_{101} x X

For each of them either give an example, or state which theorem you are using to explain why such graphs do not exist.

By Vizing's Theorem, $\Delta \leq \chi'(G) \leq \Delta+1$ where Δ is the largest vertex degree of G.

a) $90 < 100$, so by Vizing's theorem the graph does not exist.

b) The tree with 101 vertices, 100 of which are endpoints, will have a chromatic index of 100 and largest vertex degree 100.

c) K_{101} will have largest vertex degree 100 and $\chi'(K_{101}) = 101$ since 101 is odd.

d) $103 > 101$, so by Vizing's Theorem the graph does not exist

e) $122 > 101$, so by Vizing's theorem the graph does not exist.

16. (10 points) Does there exist a bipartite graph with largest vertex degree 1000 and chromatic index 1001?. State the name of the theorem that you are invoking.

No. By König's Theorem, a bipartite graph with largest degree Δ has chromatic index Δ, so the chromatic index must be 1000

6sides

17. (10 points) A simple planar graph with n vertices only has $\boxed{\text{hexagonal}}$ faces. How many edges does it have? Can n be odd?

let $m =$ # of edges, $f =$ # of faces. $n-m+f=2$ by Euler's formula

Since every side is a hexagon, $6f=2m$, $f=(2/6)m$.

Then $n-m+(2/6)m=2$, $n-(4/6)m=2$, $(-2/3)m=2-n$, $(2/3)m=n-2$,

$\boxed{m=\frac{3n-6}{2} \text{ edges}}$

If n is odd, $n=2K+1$. $m=\frac{3(2K+1)-6}{2}=\frac{6K+3-6}{2}=\frac{6K-3}{2}=3K-\frac{3}{2}$

where K is a positive integer

The graph cannot have $3K-3/2$ edges, so n must be even.

18. (10 points) Prove that every simple planar graph contains a vertex of degree at most 5.

Scratch
$6n \leq 2m$, $m \geq 3n$
$3f \leq 2m$
$3f \leq 6n$
$n-m+f \leq$
$m \leq 3n-6$

Proof by Contradiction: Suppose every vertex of a simple planar graph has degree at least 6.

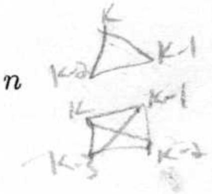
Then by the Handshaking Lemma, $6n \leq 2m$ where n is the number of vertices and m is the number of edges. Thus, $3n \leq m$.

However, since the upper bound for the number of edges in a simple planar graph is $3n-6$, $m \leq 3n-6$ must be true.

Then $3n \leq m \leq 3n-6$. This is a contradiction as since n is non-negative, $3n > 3n-6$.

Therefore, a simple planar graph must have a vertex of degree at most 5. $\square$

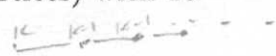
19. (10 points) What is the chromatic polynomial of (i) K_n (the complete graph on n vertices) (ii) N_n (the null graph in n vertices)



i) $P_{K_n}(K) = K(K-1)\cdots\cdots(K-n+1)$

ii) $P_{N_n}(K) = K^n$

20. (10 points) In how many ways can you color P_{10} (the path with 10 vertices) with 10 colors?



$P_{P_n}(K) = K(K-1)^{n-1}$

$\boxed{P_{P_{10}}(10) = 10(9)^9 = 3{,}874{,}004{,}890 \text{ ways}}$

$9^9 = 9^2 \cdot 9^2 \cdot 9^2 \cdot 9^2 \cdot 9 = 81 \cdot 81 \cdot 81 \cdot 81 \cdot 9 = 6561 \cdot 6561 \cdot 9$

$= 43046721 \cdot 9$

$= 387{,}400{,}489$

$$\begin{array}{r} 81 \\ \times 81 \\ \hline 81 \\ +6480 \\ \hline 6561 \end{array} \qquad \begin{array}{r} 6561 \\ \times 6561 \\ \hline 6561 \\ 393660 \\ 3280500 \\ +39366000 \\ \hline 43046721 \end{array}$$

$$\begin{array}{r} 43046721 \\ \times \quad 9 \\ \hline 387{,}400{,}489 \end{array}$$