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Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Bonnie Amende, Paul Bracken, Hongwei Chen, Raluca Dumitru, Zachary Franco, George Gilbert, Leonid V. Kovalev, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at
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Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by July 31, 2026. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12587. *Proposed by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy.* A partition of a positive integer n may be viewed as a strictly increasing list of positive integers $k_1, \dots, k_r$ and a list of positive integers $m_1, \dots, m_r$ with $m_1k_1 + \cdots + m_rk_r = n$. Let $f(n) = \sum(-1)^{m_1+\cdots+m_r-r} m_1 \cdots m_r$, where the sum is over all partitions of n. For example, if $n = 4$, then the partitions are $4 \cdot 1$, $2 \cdot 1 + 1 \cdot 2$, $2 \cdot 2$, $1 \cdot 1 + 1 \cdot 3$, and $1 \cdot 4$, and therefore $f(4) = -4 - 2 \cdot 1 - 2 + 1 \cdot 1 + 1 = -6$. Show that if n is not the sum of two squares, then 5 divides $f(n)$.

12588. *Proposed by H. A. ShahAli, Tehran, Iran.* A *flip* of a $\{0, 1\}$-matrix is the result of choosing a row and flipping all its entries: changing 0 to 1 and 1 to 0. Call a matrix M *alternating* if for all sequences $M_1, M_2, \dots$ with $M_1 = M$ and with M_{k+1} a flip of M_k for all $k \geq 1$, the matrices in the sequence alternate between having a column of zeros and not having a column of zeros. Determine all pairs of positive integers m, n such that there exists an alternating m-by-n matrix.

12589. *Proposed by Robert Rogojan, Baia Mare, Romania, and George Ţurcaş, Cluj-Napoca, Romania.* Let G be a finite abelian group with at least three elements. For any group H, let $p(H)$ be the product of all elements of H. Let $s(G)$ be the number of distinct elements $p(H)$ as H varies over all subgroups of G distinct from $\{e\}$ and G, and let $n(G)$ equal the number of elements of G of order 2.
(a) Prove that $n(G) \leq s(G) \leq n(G) + 1$
(b) Find all fiinite abelian groups G such that $s(G) = n(G)$.

12590. *Proposed by Hongwei Lou and Jinjai Yan, Fudan University, Shanghai, China.* Evaluate

$$\lim_{x \to 0^+} x \ln x \sum_{n=2}^{\infty} \frac{\sin(nx)}{\ln n}.$$

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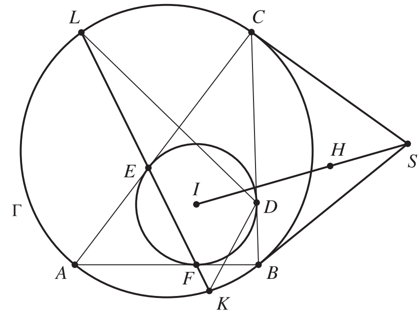
12591. *Proposed by Dong Luu, Hanoi National University of Education, Hanoi, Vietnam.* Let $\triangle ABC$ be a triangle with $\angle A \neq 90°$. Let $\triangle ABC$ have circumcircle Γ and incenter I. Let D, E, F be the tangency points of the incircle with BC, CA, AB, respectively. Let K and L be the intersection points of the line through E and F with Γ, and let the tangents to Γ at B and C meet at S. Prove that H, the orthocenter of $\triangle DKL$, lies on the line SI.



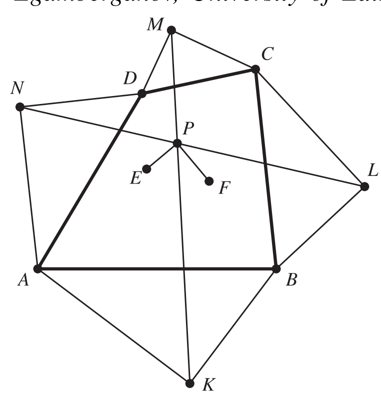
12592. *Proposed by Elliot Glazer, Principia Labs, San Francisco, CA.* Consider a biased coin C with sides labeled 0 and 1. When C is flipped, it shows 0 with unknown probability p and 1 with probability $1 - p$, where $0 < p < 1$. Let the *critical value* of a nonconstant bit sequence be the number of bits in the maximal constant string starting at the beginning.
(a) Show how to use C to simulate a fair coin by a method that assigns to any bit sequence with critical value n a result after at most $2n$ flips of C. For example, if the first five flips of C are 00001, then $n = 4$ and the method must declare heads or tails after at most three more flips.
(b) Find a simulation method as in part **(a)** that always gives a result after $\max(2, 2n - 1)$ flips of C, where n is the critical value.

12593. *Proposed by Rodney Nillsen, University of Wollongong, Wollongong, Australia.* For a sequence $w_1, w_2, \ldots$ of positive integers, let x be the real number whose binary representation is $0.0^{w_1}1^{w_2}0^{w_3}1^{w_4}\ldots$, where d^{w_n} denotes the string consisting of the digit d repeated w_n times. Show that if $\limsup w_{j+1}/\sum_{k=1}^{j} w_k > 1$, then x is transcendental.

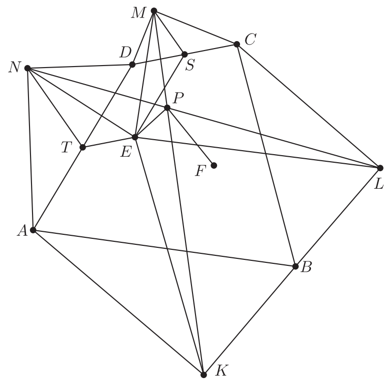
SOLUTIONS

A Generalization of Van Aubel's Theorem

12466 [2024, 446]. *Proposed by Khakimboy Egamberganov, University of Edinburgh, Edinburgh, UK.* Let $ABCD$ be a convex quadrilateral and let AKB, BLC, CMD, DNA be similar right-angled triangles constructed externally to $ABCD$, where $\angle AKB = \angle BLC = \angle CMD = \angle DNA = 90°$ and $\angle KAB = \angle LCB = \angle MCD = \angle NAD$. Let E and F bisect the diagonals AC and BD, respectively, and let P be the intersection of KM and LN. Prove that $\angle EPF$ is a right angle.



Solution I by the proposer. Let $\theta = \angle MCD$, and let S and T be the midpoints of CD and DA, respectively. Since $SDTE$ is a parallelogram and since $\angle DNA = \angle CMD = 90°$, we have $SE = DT = TN$ and $SM = DS = TE$. Also, $\angle MSE = 2\theta + \pi - \angle SET = \angle ETN$, so triangles MSE and ETN are congruent, and therefore $EM = EN$. Furthermore, $\angle MEN = \angle SET - (\pi - \angle MSE) = 2\theta$. Similarly, $EK = EL$ and $\angle KEL = 2\theta$. Therefore $\triangle ELN$ is a rotation of $\triangle EKM$ by 2θ around E. Consequently, the altitudes from E to LN and KM are equal, which implies that E is on the bisector of $\angle NPK$. Likewise, F is on the bisector of $\angle KPL$, and hence $EP \perp FP$.



Solution II by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. We lay down complex coordinates with the midpoint of EF as the origin, and identify a point with its complex coordinate. Since $E = (A + C)/2$ and $F = (B + D)/2$, we have $A + B + C + D = 0$. Point K lies on the semicircle with diameter AB, so we have $K = ((A + B) + e^{it}(A - B))/2$ for some $t \in (0, \pi)$. We then have

$$L = \frac{C+B+e^{-it}(C-B)}{2}, M = \frac{C+D+e^{it}(C-D)}{2}, \text{ and } N = \frac{A+D+e^{-it}(A-D)}{2}.$$

It suffices to prove that P lies on the circle Γ with center 0 and radius $|E|$. Notice that the midpoints P_1 of KM and P_2 of LN are both on Γ, with $P_1 = e^{it}(A + C)/2 = e^{it}E$ and $P_2 = e^{-it}(A + C)/2 = e^{-it}E$, and the central angle between P_1 and P_2 is $2t$.

We now compute the angle between the lines KM and LN. We have

$$K - M = A + B + e^{it}(A + D) = e^{it}(A + D + e^{-it}(A + B)) = e^{it}(N - L),$$

so this angle is t. This is also the angle between PP_1 and PP_2, and hence P is on Γ.

Editorial comment. Lossers, Roberto Tauraso, and the Davis Problem Solving Group observed that this configuration generalizes Van Aubel's theorem, in which AKB is an isosceles right triangle.

Also solved by M. Bataille (France), K. Gatesman, J.-P. Grivaux (France), K.-W. Lau (China), C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), and Davis Problem Solving Group.

Cut Me a Little Slack

12468 [2024, 536]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* With $[n] = \{1, \dots, n\}$, define the s-slack of a permutation $[a_1, \dots, a_n]$ of $[n]$ to be

$$\min_{1 \le k \le n+1-s} \max\{a_k, \dots, a_{k+s-1}\}.$$

For example, the permutation [3, 1, 4, 5, 9, 2, 6, 8, 7] has s-slacks (1, 3, 4, 5, 9, 9, 9, 9, 9) for s from 1 to 9, respectively. Let $S_n(s, t)$ be the number of permutations of $[n]$ whose s-slack equals t. For example, $S_4(2, 2) = S_4(2, 3) = S_4(3, 3) = S_4(3, 4) = 12$.

(a) Show that $S_n(s, t)$ can be computed exactly by performing $O\left(n + t^2\right)$ arithmetic operations on integers that are at most $n!$.

(b) Find constants A and B such that the expected 2-slack of a random permutation of $[n]$ is asymptotic to An^B.

(c)* What is the asymptotic behavior of the expected s-slack of a random permutation of $[n]$?

Solution to part **(a)** *by Kyle Gatesman, Fairfax, VA.* Let $G_n(s,t)$ be the number of permutations of $[n]$ whose s-slack is greater than t. Thus

$$S_n(s,t) = G_n(s,t-1) - G_n(s,t),$$

and it suffices to show that we can compute $G_n(s,t)$ by performing $O\left(n+t^2\right)$ arithmetic operations on integers that are at most $n!$.

Let $A_n(s,t)$ be the number of binary strings of length n having t zeros and $n-t$ ones and having no run of s consecutive zeros. For a permutation $[a_1,\dots,a_n]$ of $[n]$, we can build a string of t zeros and $n-t$ ones by putting a zero in position i if $1 \le a_i \le t$ and a one in position i if $t < a_i \le n$. We call this string the *image* of the permutation. Separately permuting the elements that are at most t and the elements that exceed t does not change the image. Hence each such image arises from exactly $t!(n-t)!$ permutations.

Furthermore, a permutation is counted by $G_n(s,t)$ if and only if its image is counted by $A_n(s,t)$. Suppose that the image has s consecutive zeros from position k to position $k+s-1$. These s entries in the permutation are all at most t. Hence $\max\{a_k,\dots,a_{k+s-1}\} \le t$, yielding s-slack at most t. Conversely, if the s-slack is at most t, then some s consecutive elements are all at most t, yielding s consecutive zeros in the image. Thus

$$G_n(s,t) = t!(n-t)!A_n(s,t).$$

Since computing $t!(n-t)!$ requires $O(n)$ operations, it suffices to show that $A_n(s,t)$ can be computed by performing $O\left(t^2\right)$ arithmetic operations on integers that are at most $n!$. Since $A_n(s,t) \le 2^n$, the size restriction is easily checked in the argument below, and we will not discuss it further.

Given any string of t zeros and $n-t$ ones, we can view the ones as cutting the string into $n-t+1$ blocks, each consisting of a (possibly empty) string of zeros. We want to count the strings in which each block contains fewer than s zeros. For $1 \le k \le n-t+1$, let A_k be the set of strings of t zeros and $n-t$ ones in which the kth block contains at least s zeros. A string is counted by $A_n(s,t)$ if and only if it is in the complement of $\bigcup_k A_k$. The principle of inclusion–exclusion yields

$$A_n(s,t) = \sum_{S\subseteq[n-t+1]} (-1)^{|S|} \left|\bigcap_{j\in S} A_j\right| = \sum_{j\ge 0} (-1)^j \binom{n-t+1}{j}\binom{n-js}{t-js}.$$

There are at most $t+1$ nonzero terms in this sum, so it suffices to show that each summand can be computed in $O(t)$ operations. This is true since the binomial coefficient $\binom{m}{k}$ equals $m(m-1)\cdots(m-k+1)/k!$, which we can compute in at most $O(k)$ operations.

Solution to part **(b)** *by the proposer.* Let $X_{s,n}$ denote the s-slack of a random permutation of $[n]$. We prove $E[X_{2,n}] \sim \sqrt{\pi n}/2$.

A permutation with s-slack exactly k is counted in $G_n(s,t)$ if and only if $0 \le t < k$, so it contributes k times to the middle sum below (we may extend the upper limit to infinity even though only finitely many terms are nonzero). Thus

$$E[X_{s,n}] = \sum_{k=1}^{\infty} \frac{kS_n(s,k)}{n!} = \sum_{t=0}^{\infty} \frac{G_n(s,t)}{n!} = \sum_{t=0}^{\infty} \frac{A_n(s,t)}{\binom{n}{t}}.$$

A string counted by $A_n(2,t)$ arises from a string of $n-t$ ones by placing zeros in t distinct locations among the $n-t+1$ locations between or outside the ones. Therefore

$$A_n(2,t) = \binom{n-t+1}{t},$$

and hence

$$E[X_{2,n}] = \sum_{t=0}^{\lfloor (n+1)/2 \rfloor} \frac{\binom{n-t+1}{t}}{\binom{n}{t}}.$$

The summand here is a decreasing function of t, is exponentially small when $t \geq n^{1/2+\epsilon}$, and Stirling's formula gives

$$\frac{\binom{n-t+1}{t}}{\binom{n}{t}} = e^{-t^2/n} + O(n^{-1/2+3\epsilon}),$$

for $t \leq n^{1/2+\epsilon}$. Therefore

$$E[X_{2,n}] = \sum_{t=0}^{\infty} e^{-t^2/n} + O(n^{4\epsilon}).$$

Since this sum is $\sqrt{n}$ times a Riemann sum for $\int_0^\infty e^{-x^2}\,dx$, which equals $\sqrt{\pi}/2$, we obtain $E[X_{2,n}] \sim \sqrt{\pi n}/2$. A more complete analysis of this sum can be found in the discussion leading up to Equation (9.94) from R. L. Graham, D. E. Knuth, and O. Patashnik (1988), *Concrete Mathematics*, Addison–Wesley.

Solution to part **(c)** *by the editors.* Consider the generating function for binary strings without s consecutive zeros in which the exponent on x records the number of zeros and the exponent on y records the total length:

$$\mathcal{A}_s(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_n(s,k)x^k y^n.$$

Letting m denote the number of ones in the string, we have

$$\mathcal{A}_s(x,y) = \sum_{m=0}^{\infty} y^m \sum_{k=0}^{\infty} A_{k+m}(s,k)(xy)^k. \qquad (*)$$

In the inner sum, the number of ones has the fixed value m, so the binary strings involved in the sum have $m+1$ blocks of zeros, each having length less than s. If we let k_j denote the number of zeros in block j for $1 \leq j \leq m+1$, then we can add up the contributions of all strings by running through all possible values of k_1 through k_{m+1}. Hence

$$\sum_{k=0}^{\infty} A_{k+m}(s,k)(xy)^k = \sum_{k_1=0}^{s-1}\sum_{k_2=0}^{s-1}\cdots\sum_{k_{m+1}=0}^{s-1}(xy)^{k_1+\cdots+k_{m+1}}$$

$$= \sum_{k_1=0}^{s-1}(xy)^{k_1}\sum_{k_2=0}^{s-1}(xy)^{k_2}\cdots\sum_{k_{m+1}=0}^{s-1}(xy)^{k_{m+1}} = \left(\sum_{k=0}^{s-1}(xy)^k\right)^{m+1} = \left(\frac{1-(xy)^s}{1-xy}\right)^{m+1}.$$

Plugging that into $(*)$ yields

$$\mathcal{A}_s(x,y) = \sum_{m=0}^{\infty} y^m \left(\frac{1-x^s y^s}{1-xy}\right)^{m+1} = \frac{1-x^s y^s}{1-y-xy+x^s y^{s+1}},$$

where we have used the formula for the sum of a geometric series to get the final closed form. Setting $x = t/(1-t)$ and $y = (1-t)y$ throughout and simplifying, we obtain

$$\mathcal{A}_s\left(\frac{t}{1-t},(1-t)y\right)=\sum_{k,n}A_n(s,k)t^k(1-t)^{n-k}y^n=\frac{1-t^sy^s}{1-y+t^s(1-t)y^{s+1}}.$$

When integrating the generating function, we recognize the Beta integral

$$\int_0^1 t^k(1-t)^{n-k}\,dt=\frac{k!(n-k)!}{(n+1)!}=\frac{1}{(n+1)\binom{n}{k}}.$$

Thus

$$\int_0^1 \mathcal{A}_s\left(\frac{t}{1-t},(1-t)y\right)dt=\int_0^1\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_n(s,k)t^k(1-t)^{n-k}y^n dt$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{A_n(s,k)}{(n+1)\binom{n}{k}}y^n.$$

As observed in part **(b)**, $E[X_{s,n}]=\sum_{k=0}^{n}A_n(s,k)/\binom{n}{k}$. Putting this all together yields

$$\sum_{n=0}^{\infty}\frac{E[X_{s,n}]}{n+1}y^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{A_n(s,k)}{(n+1)\binom{n}{k}}y^n=\int_0^1\mathcal{A}_s\left(\frac{t}{1-t},(1-t)y\right)dt$$
$$=\int_0^1\frac{1-t^sy^s}{1-y+t^s(1-t)y^{s+1}}\,dt.$$

From this integral representation, the generating function is analytic for $|y|<1$. The dominant singularity on the unit circle comes when $y\to 1$. Substituting $t=uz$ and $y=1-u^s$ (so that $u\to 0$ as $y\to 1$), the integral becomes

$$u^{1-s}\int_0^{1/u}\frac{1-u^s(1-u^s)^sz^s}{1+z^s(1-uz)(1-u^s)^{s+1}}\,dz.$$

This integral converges as $u\to 0^+$, yielding

$$\sum_{n=0}^{\infty}\frac{E[X_{s,n}]}{n+1}y^n\sim u^{1-s}\int_0^{\infty}\frac{1}{1+z^s}\,dz=(1-y)^{-1+1/s}\Gamma(1+1/s)\Gamma(1-1/s).$$

The integral here is a beta integral after a substitution; see formula 3.241.2 (with $\mu=1$ and $\nu=s$) from I. S. Gradshteyn and I. M. Ryzhik (2015), *Table of Integrals, Series, and Products, 8th Edition*, Academic Press.

The coefficient of y^n in the power series of $(1-y)^{-1+1/s}$ is

$$\binom{n-1/s}{n}=\frac{1}{\Gamma(1-1/s)}n^{-1/s}(1+O(1/n)),$$

so

$$E[X_{s,n}]\sim\Gamma(1+1/s)n^{1-1/s}.$$

Editorial comment. The analysis in the solution to part **(c)** can be carried out to higher order by expanding the integral as a series in u. This shows that $E[X_{s,n}]$ has an asymptotic expansion as a series in powers of $n^{1/s}$. For example, carrying out the analysis until the error term is small as $n\to\infty$, we get

$$E[X_{2,n}]=\frac{1}{2}\sqrt{\pi n}+\frac{1}{2}+O(n^{-1/2})$$

(which was derived by the proposer, though with a larger error bound) and

$$E[X_{3,n}] = \Gamma(4/3)n^{2/3} + \frac{\Gamma(5/3)}{3}n^{1/3} + \frac{1}{3} + O(n^{-1/3}).$$

Also solved by H. von Eitzen (Germany). Parts **(a)** and **(b)** solved by K. Gatesman and the proposer.

A Logarithmic Hyperbolic Tangent Series

12470 [2024, 536]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(\frac{\tanh(2^n)}{\tanh\left(2^{n-1}\right)}\right).$$

Solution by Lixing Han, University of Michigan, Flint, MI. The desired sum is $\ln(1 + e^{-2})$. Since

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{(1 - e^{-2x})^2}{1 - e^{-4x}},$$

for the desired sum we compute

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(\frac{\tanh(2^n)}{\tanh\left(2^{n-1}\right)}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\ln\left(\frac{(1 - e^{-2^{n+1}})^2}{1 - e^{-2^{n+2}}}\right) - \ln\left(\frac{(1 - e^{-2^n})^2}{1 - e^{-2^{n+1}}}\right)\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n}\left(2\ln(1 - e^{-2^{n+1}}) - \ln(1 - e^{-2^{n+2}}) - 2\ln(1 - e^{-2^n}) + \ln(1 - e^{-2^{n+1}})\right)$$

$$= \sum_{n=1}^{\infty} \frac{\ln(1 - e^{-2^{n+1}}) - \ln(1 - e^{-2^n})}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{\ln(1 - e^{-2^{n+2}}) - \ln(1 - e^{-2^{n+1}})}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{\ln(1 - e^{-2^{n+2}}) - \ln(1 - e^{-2^{n+1}})}{2^n} - \sum_{n=1}^{\infty} \frac{\ln(1 - e^{-2^{n+2}}) - \ln(1 - e^{-2^{n+1}})}{2^n}$$

$$= \frac{\ln(1 - e^{-4}) - \ln(1 - e^{-2})}{2^0} = \ln\left(\frac{1 - e^{-4}}{1 - e^{-2}}\right) = \ln(1 + e^{-2}).$$

Also solved by T. Amdeberhan, C. P. Anil Kumar (India), M. Bataille (France), H. Chen (US), B. E. Davis, H. von Eitzen (Germany), G. Fera (Italy), K. Gatesman, M. L. Glasser, N. Grivaux (France), E. A. Herman, N. Hodges (UK), E. J. Ionascu, W. Janous (Austria), S. Kaczkowski, K. T. L. Koo (China), O. Kouba (Syria), T. Koupelis, B. Lai (China), P. Lalonde (Canada), K.-W. Lau (China), O. P. Lossers (Netherlands), J. Magliano, D.-S. Marinescu & M.-I. Marinescu (Romania), R. E. Maza (Philippines), R. Mortini (Luxembourg) & R. Rupp (Germany), R. Nandan, H. Ohtsuka (Japan), P. Perfetti (Italy), C. R. Pranesachar (India), V. Schindler (Germany), S. Seto, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, Y. Zhang (China), and the proposer.

Surprises in an Isosceles Tetrahedron

12471 [2024, 537]. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* A tetrahedron is *isosceles* if all pairs of opposite edges have equal length. Let $SABC$ be an isosceles tetrahedron, and let X, Y, and Z be points on segments SA, SB, and SC, respectively, such that $SX = SY = SZ$. Let G be the centroid of $\triangle XYZ$. Prove $3\,SG = SX\sqrt{5 + 2r/R}$, where R and r are the circumradius and inradius of $\triangle ABC$, respectively.

Solution by Eugen J. Ionascu, Columbus State University, Columbus, GA. Because the problem is invariant under rescaling, we may assume $SX = SY = SZ = 1$. Since G is the centroid of $\triangle XYZ$, we have $\overrightarrow{SG} = (\overrightarrow{SX} + \overrightarrow{SY} + \overrightarrow{SZ})/3$, and therefore

$$SG^2 = \overrightarrow{SG} \cdot \overrightarrow{SG} = \frac{3 + 2\overrightarrow{SX} \cdot \overrightarrow{SY} + 2\overrightarrow{SX} \cdot \overrightarrow{SZ} + 2\overrightarrow{SY} \cdot \overrightarrow{SZ}}{9}.$$

Let α, β, and γ be the angles of $\triangle ABC$ at the vertices A, B, and C, respectively. Since the tetrahedron is isosceles, $\triangle SBC \cong \triangle ACB$. Therefore $\angle YSZ = \angle BSC = \alpha$, so $\overrightarrow{SY} \cdot \overrightarrow{SZ} = \cos\alpha$. Similarly, $\overrightarrow{SX} \cdot \overrightarrow{SZ} = \cos\beta$ and $\overrightarrow{SX} \cdot \overrightarrow{SY} = \cos\gamma$, so we have

$$9\, SG^2 = 3 + 2\cos\alpha + 2\cos\beta + 2\cos\gamma. \tag{1}$$

We next use

$$\cos\alpha + \cos\beta + \cos\gamma = 1 + \frac{r}{R}. \tag{2}$$

This follows from Carnot's theorem, but for completeness we give a self-contained derivation below. Combining (1) and (2), we obtain $9\, SG^2 = 5 + 2r/R$, which yields the desired result.

To prove (2), let a, b, and c be the lengths of the sides of $\triangle ABC$ opposite the vertices A, B, and C, respectively, and let T be the area of $\triangle ABC$. We use three well-known formulas for T:

$$T = \frac{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}}{4}, \quad T = \frac{r(a+b+c)}{2}, \quad \text{and} \quad T = \frac{abc}{4R}.$$

Applying the law of cosines, we get

$$\cos\alpha + \cos\beta + \cos\gamma - 1 = \frac{b^2a + c^2a - a^3 + a^2b + c^2b - b^3 + a^2c + b^2c - c^3 - 2abc}{2abc}$$
$$= \frac{(a+b-c)(b+c-a)(c+a-b)}{2abc} = \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{2abc(a+b+c)}$$
$$= \frac{16T^2}{2(4RT)(2T/r)} = \frac{r}{R}.$$

Also solved by C. P. Anil Kumar (India), M. Bataille (France), H. Chen (China), P. De (India), I. Dimitrić, G. Fera (Italy), K. Gatesman, N. Hodges (UK), W. Janous (Austria), K. T. L. Koo (China), O. Kouba (Syria), O. P. Lossers (Netherlands), D.-S. Marinescu & M.-I. Marinescu (Romania), C. Petalas (Greece), C. R. Pranesachar (India), V. Schindler (Germany), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Zhou, Davis Problem Solving Group, and the proposer.

Integers in a Normalized Recurrence

12472 [2024, 537]. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden.* Let $a_1 = 2$, $a_2 = 3$, and $a_n = 2n + a_{\lfloor (n-1)/2 \rfloor} + a_{\lceil (n-1)/2 \rceil}$ for $n \geq 3$. Which integers equal a_n/n for some value of n?

Solution by Yuri J. Ionin, Central Michigan University, Mount Pleasant, MI. The integers equal to a_n/n for some value of n are 2 and the positive integers congruent to 8 modulo 10.

Lemma 1. *Let $d_n = a_n - a_{n-1}$ for $n \geq 2$. This sequence satisfies $d_{2n} = d_{2n+1} = 2 + d_n$ for $n \geq 2$.*

Proof. We compute

$$d_{2n} = a_{2n} - a_{2n-1} = (4n + a_{n-1} + a_n) - (4n - 2 + 2a_{n-1}) = 2 + d_n$$
$$d_{2n+1} = a_{2n+1} - a_{2n} = (4n + 2 + 2a_n) - (4n + a_{n-1} + a_n) = 2 + d_n,$$

as claimed. ■

Lemma 2. *For $k \geq 1$,*

$$d_n = \begin{cases} 2k - 1 & \text{if } 2^k \leq n < 2^k + 2^{k-1}, \\ 2k + 5 & \text{if } 2^k + 2^{k-1} \leq n < 2^{k+1}. \end{cases}$$

Proof. We use induction on k. For $k = 1$, we have $d_2 = 1$ and $d_3 = 7$. For $k \geq 2$, consider $n \in [2^k, 2^{k+1})$. By Lemma 1, $d_n = 2 + d_{\lfloor n/2 \rfloor}$. By the induction hypothesis, $d_n = 2 + 2(k-1) - 1 = 2k - 1$ when $n \in [2^k, 2^k + 2^{k-1})$ and $d_n = 2 + 2(k-1) + 5 = 2k + 5$ when $n \in [2^k + 2^{k-1}, 2^{k+1})$. ■

Lemma 3. *For $k \geq 1$,*

(1) $a_{2^k-1} = 2 + (k-1)2^{k+1}$,
(2) $a_{2^k+m} = a_{2^k-1} + (m+1)(2k-1)$ *for* $0 \leq m < 2^{k-1}$,
(3) $a_{2^k+2^{k-1}+\ell} = a_{2^k+2^{k-1}-1} + (\ell+1)(2k+5)$ *for* $0 \leq \ell < 2^{k-1}$.

Proof. We prove (1) by induction on k. For the basis, $a_1 = 2$. Now consider $k \geq 2$. Using the definition of d_n to "untelescope" $a_{2^k-1} - a_{2^{k-1}-1}$, followed by Lemma 2 and later the induction hypothesis, we compute

$$\begin{aligned} a_{2^k-1} &= a_{2^{k-1}-1} + \sum_{n=2^{k-1}}^{2^k-1} d_n = a_{2^{k-1}-1} + \sum_{n=2^{k-1}}^{2^{k-1}+2^{k-2}-1} (2k-3) + \sum_{n=2^{k-1}+2^{k-2}}^{2^k-1} (2k+3) \\ &= a_{2^{k-1}-1} + (2k-3)2^{k-2} + (2k+3)2^{k-2} = a_{2^{k-1}-1} + k2^k \\ &= 2 + (k-2)2^k + k2^k = 2 + (k-1)2^{k+1}. \end{aligned}$$

Statements (2) and (3) are immediate from $m + 1$ and $\ell + 1$ applications of Lemma 2, respectively. ■

For $k \in \mathbb{N}$, partition the segment $\{2^k, \ldots, 2^{k+1} - 1\}$ of integers into A_k and B_k, where $A_k = \{2^k, \ldots, 2^k + 2^{k-1} - 1\}$ and $B_k = \{2^k + 2^{k-1}, \ldots, 2^{k+1} - 1\}$. Every integer that is at least 2 belongs to A_k or B_k for some positive integer k.

Lemma 4. *Let $n \in \mathbb{N}$.*

(1) If $n \in A_k$ for some k, then $a_n/n \notin \mathbb{Z}$.
(2) If $n \in B_k$ for some k, then $a_n/n = 2k + 5 - \left(5(2^{k+1}) - 2k - 7\right)/n$.

Proof. For (1), note first that $a_n/n \in \mathbb{Z}$ requires $a_n \equiv 0 \pmod{n}$. For $n \in A_k$, we write $n = 2^k + m$ where $0 \leq m < 2^{k-1}$. Applying Lemma 3 and reducing 2^k to $-m$ modulo n, we have

$$\begin{aligned} a_n &= 2 + (k-1)2^{k+1} + (m+1)(2k-1) \\ &\equiv 2 - 2m(k-1) + (m+1)(2k-1) \pmod{n} = m + 2k + 1. \end{aligned}$$

If $k = 1$ or $k = 2$, then $2k = 2^k$, which yields $a_n \equiv 1 \not\equiv 0 \pmod{n}$. If $k > 2$, then $2k + 1 < 2^k$, and hence $m + 2k + 1$ is already reduced modulo n and is not congruent to 0. Therefore $a_n/n \notin \mathbb{Z}$ for $n \in A_k$.

For (2), consider $n \in B_k$, letting $n = 2^k + 2^{k-1} + \ell$ with $0 \leq \ell < 2^{k-1}$. By Lemma 3, with $m = 2^{k-1} - 1$,

$$\begin{aligned} a_n &= a_{2^k+m} + (\ell+1)(2k+5) = \big(a_{2^k-1} + 2^{k-1}(2k-1)\big) + (\ell+1)(2k+5) \\ &= \big((2+(k-1)2^{k+1}) + 2^{k-1}(2k-1)\big) + (\ell+1)(2k+5) \\ &= 2 + (6k-5)2^{k-1} + (n - 2^k - 2^{k-1} + 1)(2k+5) \\ &= (2k+5)n - \big(5(2^{k+1}) - 2k - 7\big). \end{aligned}$$ ■

Lemma 5. *If $n \in B_k$ and $a_n/n \in \mathbb{Z}$, then $k \equiv 4 \pmod 5$ and $a_n/n = 2k$.*

Proof. Lemma 4 yields $a_n/n = 2k + 5 - r$, where $r = (5(2^{k+1}) - 2k - 7)/n$. Thus $r \in \mathbb{Z}$. We may assume $k \geq 3$, because $r = 11/3$ when $k = 1$ and $r \in \{29/6, 29/7\}$ when $k = 2$. When $k \geq 3$, we have $2^{k+1} > 2k + 7$; also note $3(2^{k-1}) \leq n < 2^{k+1}$. Hence

$$4 = \frac{5(2^{k+1}) - 2^{k+1}}{2^{k+1}} < r < \frac{5(2^{k+1})}{3(2^{k-1})} = \frac{20}{3} < 7.$$

As an integer, r must be odd because its numerator is odd. Hence $r = 5$. From $a_n/n = 2k + 5 - r$ we therefore obtain $a_n/n = 2k$. Also $5(2^{k+1}) - 2k - 7 = 5n$, which implies $k \equiv 4 \pmod 5$. ■

When $n = 1$, $a_n/n = 2$. Lemmas 4 and 5 show that every integer greater than 2 having the form a_n/n for some $n \in \mathbb{N}$ is congruent to 8 modulo 10. Conversely, it remains only to prove that $10h + 8$ has this form for $h \geq 0$. Let $s = 10h + 8$, and set $k = s/2 = 5h + 4$, $\ell = 2^{k-1} - 2h - 3$, and $n = 2^k + 2^{k-1} + \ell$, so $n \in B_k$. By Lemma 4, $a_n/n = (2k+5) - r$, where $r = \big(5(2^{k+1}) - 2k - 7\big)/n$. For these values of k and n,

$$r = \frac{5(2^{5h+5}) - 10h - 15}{2^{5h+5} - 2h - 3} = 5.$$

Thus $s = 2k = a_n/n$, completing the proof.

Also solved by M. Aassila (France), C. P. Anil Kumar (India), B. Benesh, H. Chen (China), H. von Eitzen (Germany), K. Gatesman, O. Geupel (Germany), N. Hodges (UK), P. Jiradilok (Thailand), O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Molinari, M. Omarjee (France), A. Stenger, R. Tauraso (Italy), and the proposer.

CLASSICS

C43. *Due to Samuel Beatty, suggested by Zachary Franco.* Let ϕ denote the golden ratio $(1+\sqrt{5})/2$. Show that every positive integer has either the form $\lfloor m\phi \rfloor$ for some positive integer m or the form $\lfloor m(\phi+1) \rfloor$ for some positive integer m, but not both.

Can an Integer Clock Show Equilateral Time?

C42. Can the ends of the hour, minute, and second hands of a standard clock with hands of integer length ever form an equilateral triangle?

Solution. The answer is yes. Let the lengths of the hour, minute, and second hands be h, m, and s, respectively. Assume the clock is in the complex plane, centered at 0 and oriented with the 12 on the clock on the imaginary axis. At t hours after midnight, the ends of the hands are positioned at $ihe^{-i\pi t/6}$, $ime^{-2i\pi t}$, and $ise^{-120i\pi t}$. Let $\omega = e^{i\pi/3}$. The hands form an equilateral triangle precisely when either

$$ise^{-120i\pi t} - ihe^{-i\pi t/6} = \omega(ime^{-2i\pi t} - ihe^{-i\pi t/6}) \qquad (1)$$

or

$$ise^{-120i\pi t} - ihe^{-i\pi t/6} = \omega^{-1}(ime^{-2i\pi t} - ihe^{-i\pi t/6}).$$

We find a solution (t, h, m, s) to (1) with t real and with positive integers for h, m, and s. Since $\omega^2 = \omega - 1$, equation (1) is equivalent to

$$se^{-120i\pi t} - \omega me^{-2i\pi t} + \omega^2 he^{-i\pi t/6} = 0.$$

Setting $z = e^{-i\pi t/6}$, we obtain

$$sz^{719}\omega^{-1} - mz^{11} + h\omega = 0. \tag{2}$$

We look for a solution to (2) with $z \in \mathbb{Q}[\omega]$ of unit norm, and with integers h, m, and s. Start by setting $z = (a + b\omega)/c$, where a, b, and c are integers with $a^2 + ab + b^2 = c^2$. An easy induction shows that

$$z^n = (r_n + s_n\omega)/c^n,$$

where $r_0 = 1$, $s_0 = 0$, $r_{k+1} = ar_k - bs_k$, and $s_{k+1} = (a + b)s_k + br_k$ for $k \geq 0$. Equation (2) becomes

$$s\frac{\omega^{-1}r_{719} + s_{719}}{c^{719}} - m\frac{r_{11} + s_{11}\omega}{c^{11}} + h\omega = 0$$

or

$$s(r_{719} + s_{719}) - mc^{708}r_{11} = \omega(sr_{719} + mc^{708}s_{11} - hc^{719}), \tag{3}$$

where we have used the identity $\omega^{-1} = 1 - \omega$.

We obtain a solution to (3) by setting

$$h = (r_{11} + s_{11})r_{719} + s_{11}s_{719}, \quad m = c^{11}(r_{719} + s_{719}), \quad \text{and} \quad s = c^{719}r_{11}.$$

This choice makes both sides of (3) equal to 0. These values of h, m, and s are clearly integers, but we need them to be positive. A calculation shows that the choice $a = -8$, $b = 3$, and $c = 7$ achieves this. From $z = e^{-i\pi t/6} = (3\omega - 8)/7$ we get $t = 6 + (6/\pi)\arctan(3\sqrt{3}/13)$; hence we get the desired equilateral triangle at approximately 6 hours, 43 minutes, and 34.415 seconds after midnight.

Editorial comment. The problem appeared in this MONTHLY as problem 12104 [2019, 370; 2020, 860]. It was posed by Joe Buhler, Larry Carter, and Richard Stong, and the solution above is due to the GCHQ Problem Solving Group. It was the only solution received other than that of the proposers.

A calculation shows that the formula above for h can be rewritten as $h = c^{22}r_{708}$. Canceling off a common factor of c^{11}, one gets a slightly smaller solution for h, m, and s. The proposers showed the remarkable fact that this reduced solution for h, m, and s (using the values of a, b, and c above) provides the smallest possible lengths for hands that can form an equilateral triangle. The length s of the second hand in this solution is $1984171195 \cdot 7^{708}$, a number with 608 decimal digits. The corresponding h and m are of a similar magnitude, with the ratio $h : m : s$ equal to approximately $1 : 9.35 : 9.88$, which represents a reasonable if somewhat unorthodox clock.

It is easy to see by continuity that there are many times in a day (in fact generally twice per minute) at which the ends of the hour and minute hands of the clock form an equilateral triangle with a third point that is on the ray formed by the second hand. At these times, the hands would form an equilateral triangle if the second hand were rescaled. So for every pair of positive integers h and m, one can find many real numbers s such that the resulting clock will describe an equilateral triangle at some time. What is rare, however, is that the second hand following such a rescaling has integer length.