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Author(s): Kjell Hausken and Matthias Mohr

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The value of a player in n-person games

Kjell Hausken[1], **Matthias Mohr**[2]

[1] School of Economics, Culture and Social Sciences, University of Stavanger, P.O. Box 2557 Ullandhaug, N-4091 Stavanger, Norway (e-mail: Kjell.Hausken@oks.his.no)
[2] Deutsche Bundesbank, Economics Department, Wilhelm-Epstein-Strasse 14-16, P.O. Box 10 06 02, D-60006 Frankfurt am Main, Germany (e-mail: Matthias.Mohr@bundesbank.de)

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Abstract. The article decomposes the Shapley value into a value matrix which gives the value of every player to every other player in n-person games. Element $\Phi_{ij}(v)$ in the value matrix is positive, zero, or negative, dependent on whether row player i is beneficial, has no impact, or is not beneficial for column player j. The elements in each row and in each column of the value matrix sum up to the Shapley value of the respective player. The value matrix is illustrated by the voting procedure in the European Council of Ministers 1981–1995.

1 Introduction

In this article the Shapley (1953) value is decomposed into a value matrix of which the elements can be interpreted as the value of every player i to every other player j, $i, j = 1, \ldots, n$. The Shapley value $\Phi_i(v)$ has traditionally been given many different interpretations. Four examples are his expected marginal contribution, the weighted average of his marginal contributions to the *coalition of all n players involved*, what player i can "reasonably" command to himself, or player i's "fair share" in the game (see e.g. Roth 1988:6). One interpretation of the *Shapley value matrix* $[\Phi_{ij}(v)]$ to be developed here is that it quantifies the value of player i's expected marginal contribution to *player j*,

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$i, j = 1, \ldots, n$. Player i makes an expected marginal contribution $\Phi_i(v)$ – the Shapley value – which is divided into $n \times n$ components $\Phi_{ij}(v)$, each of which gives the value of row player i's expected marginal contribution $\Phi_i(v)$ to every column player j, $i, j = 1, \ldots, n$. We show that player i's marginal contributions to all players (including himself) sum up to $\Phi_i(v)$. Hence player j is able to determine not only how valuable a game is to himself, but also how valuable each player i in the game is to himself. This allows player j to rank the value or importance of every player i (including himself) to himself in an n-component ranking list.[1] If player i is of a certain value to player j, then player j may have a certain interest in player i. Hence the value matrix can also be interpreted as an interest matrix.[2] Furthermore, if player j is interested in player i, this gives the possibility for player i to have power over player j. More specifically, if player j has a high interest in player i, while player i has a low interest in player j, then player i can be said to have power over player j. Hence the value matrix can also be interpreted as a power matrix.[3]

2 The value of a player to another player in an n-person game

Shapley (1953) proposed the following value $\Phi_i(v)$ to a player i, $i = 1, \ldots, n$, in an n-person game $\{v(S), S \subseteq N\}$ where N is the set of players, S is one of the 2^n subcoalitions of N, including the empty set, $v(S)$ is the characteristic function which assigns a value to every coalition S, and s is the number of players in S, i.e. $s = |S|$ and $n = |N|$:

$$\Phi_i(v, N) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{(s-1)!(n-s)!}{n!} (v(S) - v(S \setminus \{i\})). \tag{2.1}$$

Summing over all the coalitions S to which player i belongs, $\Phi_i(v)$ is player i's marginal contribution to coalition S, multiplied with the $(s-1)!$ different permutations of the members of coalition S aside from player i, multiplied

[1] The value matrix implicitly involves defining a player i's "value to himself." A 100% "self-sufficient" player i has a value to himself equal to the Shapley value $\Phi_{ii}(v) = \Phi_i(v)$, with no additional value of the other players. A 100% "other-dependent" player i, who is 0% "self-sufficient", has a value $\Phi_{ii}(v) = 0$ to himself, where the value of the other players to himself sum up to the Shapley value $\Phi_i(v)$. In most games player i has an "intermediate value to himself," which is interpreted as an interpolation between these two extremes.

[2] This implies defining a 100% "self-sufficient" player as having 100% "interest" in himself and "no interest" in other players. Conversely a 100% "other-dependent" player is defined to have "no interest in himself" (w.r.t. the game being played) and 100% interest in the other players.

[3] This implies defining a 100% "self-sufficient" player i as having 100% power "over himself" where no other players have "power over" player i. Conversely a 100% "other-dependent" player i has no power "over himself" while the other players together have 100% "power over" player i.

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with the $(n-s)!$ different permutations of the players not being members of coalition S, divided by the $n!$ different permutations of all the players in the grand coalition N.

We can analogously calculate player i's fair share in every subcoalition S, assuming that only the members of S are taking part in the subgame $\{v(R), R \subseteq S\}, S \subseteq N$:

$$\Phi_i(v, S) = \sum_{\substack{R \subseteq S \\ R \ni i}} \frac{(r-1)!(s-r)!}{s!}(v(R) - v(R \backslash \{i\}). \tag{2.2}$$

For convenience, let us write $\Phi_i(v, S) = \Phi_i(S)$. We define the value of player i to player j as

$$\Phi_{ij}(N) = \sum_{\substack{S \subseteq N \\ i,j \in S}} \frac{(n-s)!(s-1)!}{n!}(\Phi_j(S) - \Phi_j(S \backslash \{i\})). \tag{2.3}$$

Since $\Phi_j(S)$ is the share of $v(S)$ player j can command for himself in the game $\{v(R), R \subseteq S\}$, $\Phi_{ij}(S)$ is the weighted sum of i's marginal contributions to j's share of $v(S)$ in every $S \subseteq N$.

Theorem 2.1. *The sum over all j, $j = 1, \ldots, n$, of the value $\Phi_{ij}(N)$ of player i to each player j equals the Shapley value $\Phi_i(N)$ of player i, that is*

$$\Phi_i(N) = \sum_{j=1}^{n} \Phi_{ij}(N). \tag{2.4}$$

Proof: Since the Shapley value is efficient, it follows that

$$\begin{aligned}
\sum_{j=1}^{n} \Phi_{ij}(N) &= \sum_{j=1}^{n} \sum_{\substack{S \subseteq N \\ i,j \in S}} \frac{(n-s)!(s-1)!}{n!}(\Phi_j(S) - \Phi_j(S \backslash \{i\})) \\
&= \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!}\left(\sum_{j=1}^{n} \Phi_j(S) - \sum_{j=1}^{n} \Phi_j(S \backslash \{i\})\right) \\
&= \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S \backslash \{i\})) = \Phi_i(N).
\end{aligned} \tag{2.5}$$

$\Phi_{ij}(N)$ can be written as (the proof is given in Appendix 1)

$$\begin{aligned}
\Phi_{ij}(N) = \sum_{R \subseteq N} \frac{(r-1)!(n-r)!}{n!}((v(R) - v(R \backslash \{j\})) \\
- (v(R \backslash \{i\}) - v(R \backslash \{i, j\}))) \sum_{s=r}^{n} \frac{1}{s}.
\end{aligned} \tag{2.6}$$

It follows immediately that $\Phi_{ij}(N) = \Phi_{ji}(N)$. Hence the matrix $[\Phi_{ij}(N)]$, which denotes the value of row player i to column player j, is symmetric.

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Inserting $i = j$ into (2.3) or (2.6), where $\Phi_i(S\backslash\{i\}) = 0$, gives

$$\Phi_{ii}(N) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!} \Phi_i(S), \tag{2.7}$$

which represents the value of player i to himself. $\Phi_{ii}(N)$ is the weighted sum of the Shapley values $\Phi_i(S)$ of player i in all subcoalitions S.

$\Phi_{ij}(N)$ is similar to

$$\Psi_{ij}(N) = \sum_{R \subseteq N} \frac{(r-1)!(n-r)!}{n!} \times ((v(R) - v(R\backslash\{j\})) - (v(R\backslash\{i\}) - v(R\backslash\{i,j\}))), \tag{2.8}$$

which is easier to interpret since the weight $\sum_{s=r}^{n} \frac{1}{s}$ is absent. $\sum_{i=1}^{n} \Psi_i$ does not add up to $v(N)$, but rather to $\sum_{i=1}^{n} \Psi_i = n \cdot v(N) - \sum_{i=1}^{n} v(N\backslash\{i\})$, and Ψ_{ij} is not a *decomposition* of the Shapley value. The Ψ-Matrix, briefly discussed in Appendix 2, is introduced to help us interpret (2.6).

Example 2.1. Consider the game with $N = \{1,2,3\}$ where $v(1) = 180$, $v(2) = v(3) = v(2,3) = 0$, $v(1,2) = 360$, $v(1,3) = v(1,2,3) = 540$. The Shapley value given by (2.1) is $\Phi(v, \{1,2,3\}) = (390\ 30\ 120)^T$, where T means transposed. The value matrix $[\Phi_{ij}(\{1,2,3\})]$ giving the value of row player i to column player j is

$$\Phi_{ij} = \begin{bmatrix} 295 & 25 & 70 \\ 25 & 25 & -20 \\ 70 & -20 & 70 \end{bmatrix}. \tag{2.9}$$

The elements in each row and in each column of the value matrix $[\Phi_{ij}]$ in (2.8) sum up to the Shapley value of the respective player. A negative value indicates that i imposes costs on j as j would be better off if i did not take part in the game. Φ_{ij} can be negative since a player may benefit from removing another player from the game. Hence in (2.9), player 2 benefits from removing player 3, player 3 benefits from removing player 2, while all the other relationships between the players are mutually beneficial. To determine how a player's power is affected by the removal of another player, the Shapley values for the games with and without the other player present need to be calculated. Removing player 3 from the game in (2.9) gives $N = \{1,2\}$, $v(1) = 180$, $v(2) = 0$, $v(1,2) = 360$, which implies $\Phi_1 = 270$ and $\Phi_2 = 90$. The power or fair share of player 2 thus increases by 60, while the power of player 1 decreases by 120, as is also indicated by the positive Φ_{31} in (2.9). Note that $\Phi_{ii} < \Phi_i$, $i = 1,2,3$, in (2.9). Although this is not generally the case, it happens in (2.9) since players 2 and 3 are only mildly antagonistic and the other relationships

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are mutually beneficial. We suggest interpreting $\Phi_{ii} < \Phi_i$ as there being synergy present, in the sense that each player has a lower value to himself when he cannot rely on the synergy flowing from the joint operation with the other players, than the Shapley value which represents his fair share in the game, what he can reasonably command. $\Phi_{ii} > \Phi_i$ means that player i to a larger extent prefers operating alone, and that he has a very negative impact on at least one other player in the game.

The notion of the value of a player to another player is to our knowledge absent in the literature, though Owen (1972:76) proposes that his second order cross-derivatives $f_{ij} = \delta^2 f / \delta x_i \delta x_j$ "can be thought of as measuring, in some sense, the value of player j to player i."[4] Owen defines $f(x_1, \ldots, x_n) = \sum_{S \subset N} \{\prod_{j \in S} x_j \prod_{j \notin S} (1 - x_j)\} v(S)$ as the multilinear extension of $v(S)$ which can be thought of as "the expected value of the (as yet unformed) coalition," where x_i may be interpreted as the probability that player i joins the coalition S (p. 64). Hence f_i is "the expected marginal value of player i to the coalition which he will join, given that player j has probability x_j of being in the coalition, and assuming independence" (pp. 72–73). Integrating f_{ij} along the main diagonal, Owen (1972:77) calculates the co-value

$$q_{ij} = \int_0^1 f_{ij}(t, \ldots, t)\, dt$$
$$= \sum_{S \subset N; i, j \in S} \frac{s!(n-s-2)!}{n!} [v(S \cup \{i, j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S)].$$

Note that $f_{ii} = 0 = q_{ii}$, which means that a player has no value to himself in Owen's formalization. Owen's and our approaches are built on different philosophical foundations which are not directly comparable. Owen derivates a multilinear extension first w.r.t to the probability that player i joins a coalition and then w.r.t. to player j joining a coalition (which in some joint sense means determining the expected marginal value of players i and j to the coalition), and then intergrates along the main diagonal (which means, we believe, assigning equal probabilities $x_i = x_j = t$ to the players joining coalition S in the integration). On the other hand, our approach specifies neither marginal values nor equal probabilities of joining a coalition, but sums up what happens in all coalitions S where players i and j are jointly present, and compares this with what happens when player i is removed from the coalition. Owen presents three examples which we analyze using the approach in this article, thus comparing the two approaches through their applications.

Example 2.2. Consider the symmetric 3-person majority game $[2; 1, 1, 1]$, i.e. $v(1) = v(2) = v(3) = 0$, $v(1, 2) = v(1, 3) = v(2, 3) = v(1, 2, 3) = 1$. The

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[4] We would like to thank an anonymous referee of this journal for pointing this out to us.

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Shapley value given by (2.1) is $\boldsymbol{\Phi}(v, \{1, 2, 3\}) = (1/3\ 1/3\ 1/3)^T$, and the value matrix $[\Phi_{ij}(\{1, 2, 3\})]$ giving the value of row player i to column player j is

$$\Phi_{ij} = \begin{bmatrix} 0.2778 & 0.02778 & 0.02778 \\ 0.02778 & 0.2778 & 0.02778 \\ 0.02778 & 0.02778 & 0.2778 \end{bmatrix} \quad (2.10)$$

Observe that the players have a low but positive value to each other and a larger value to themselves. In contrast, Owen (1972:77) determines the co-values $q_{ij} = 0$, which he interprets "so on the average no pair of players help or hinder each other."

Example 2.3. Consider the 3-person market game with one seller (player 1) and two buyers (players 2 and 3), i.e. $[3; 2, 1, 1]$, i.e. $v(1) = v(2) = v(3) = v(2, 3) = 0$, $v(1, 2) = v(1, 3) = v(1, 2, 3) = 1$. The Shapley value given by (2.1) is $\boldsymbol{\Phi}(v, \{1, 2, 3\}) = (2/3\ 1/6\ 1/6)^T$. The value matrix $[\Phi_{ij}(\{1, 2, 3\})]$ giving the value of row player i to column player j is

$$\Phi_{ij} = \begin{bmatrix} 0.3889 & 0.1389 & 0.1389 \\ 0.1389 & 0.1389 & -0.1111 \\ 0.1389 & -0.1111 & 0.1389 \end{bmatrix}. \quad (2.11)$$

As we expect players 2 and 3's antagonism gets represented by the negative $\Phi_{23} = \Phi_{32} = -0.1111$. Note that player 2 sees a larger value by player 1 being present ($\Phi_{12} = 0.1389$) than he sees a disvalue by player 3 being present ($\Phi_{32} = -0.1111$). This is because player 2 cannot buy the good without player 1 selling, while player 2 may still buy the good if also player 3 is buying. Note that Owen determines the co-values $q_{12} = q_{13} = 1/2$ and $q_{23} = -1/2$ for this game.

Example 2.4. Consider the 4-person simple game with one strong player (2-person coalitions S win if $1 \in S$; all 3-person coalitions win), i.e. $[3; 2, 1, 1, 1]$, i.e. $v(1) = v(2) = v(3) = v(4) = v(2, 3) = v(2, 4) = v(3, 4) = 0$, $v(1, 2) = v(1, 2, 3) = v(1, 2, 4) = v(1, 3, 4) = v(2, 3, 4) = v(1, 2, 3, 4) = 1$. The Shapley value given by (2.1) is $\boldsymbol{\Phi}(v, \{1, 2, 3, 4\}) = (1/2\ 1/6\ 1/6\ 1/6)^T$, and the value matrix $[\Phi_{ij}(\{1, 2, 3\})]$ giving the value of row player i to column player j is

$$\Phi_{ij} = \begin{bmatrix} 0.4167 & 0.0278 & 0.0278 & 0.0278 \\ 0.0278 & 0.1389 & 0 & 0 \\ 0.0278 & 0 & 0.1389 & 0 \\ 0.0278 & 0 & 0 & 0.1389 \end{bmatrix}. \quad (2.12)$$

For this game player 1 is equally valuable to any other player 2, 3, or 4, and values 2, 3, and 4 equally, since only such a 2-person coalition gives payoff 1. On the other hand, players 2, 3, 4 assign value 0 to each other, since they are neither beneficial nor unbeneficial to each other. In contrast, Owen's co-values for this game are $q_{ij} = 0$.

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3 The value of a player in simple games: An empirical example

Define a simple game as $G = \{q; w_1, w_2, \dots, w_n\}$. The value of a coalition S in simple games is one if the weak inequality

$$\sum_{i \in S} w_i \geq q \tag{3.1}$$

is satisfied, in which case S qualifies for a majority, and zero otherwise. A player i is decisive in S if $i \in S$ and S satisfies (3.1), while $S\backslash\{i\}$ does not satisfy (3.1).

In simple games (2.6) can be interpreted as follows: The value of i to j is the weighted sum Φ_{ij}^+ of all coalitions S in which j is decisive if and only if i is a member of S, minus the weigthed sum Φ_{ij}^- of all coalitions S in which j is decisive if and only if i is not a member of S. Hence the value of a player to another player can be decomposed into the difference of the two weigthed sums

$$\Phi_{ij} = \sum_{\substack{R \subseteq N; R \ni i,j \\ j \text{ decisive in } R, \\ j \text{ not decisive in } R\backslash\{i\}}} \frac{(r-1)!(n-r)!}{n!} \sum_{s=r}^{n} \frac{1}{s} - \sum_{\substack{R \subseteq N; R \ni i,j \\ j \text{ not decisive in } R, \\ j \text{ decisive in } R\backslash\{i\}}} \frac{(r-1)!(n-r)!}{n!} \sum_{s=r}^{n} \frac{1}{s} = \Phi_{ij}^+ - \Phi_{ij}^-, \quad i \neq j. \tag{3.2}$$

The weighted sums Φ_{ij}^+ and Φ_{ij}^- can be interpreted as "positive" and "negative" power, respectively. Positive power accounts for all cases where i makes j decisive, and negative power accounts for all cases where i prevents j from being decisive, which represents the power of player i to block (i.e. to obstruct) player j. Hence Φ_{ij} is positive if and only if i's power of making j decisive is larger than i's power of blocking j.

Note that $\Phi_{ii}^- = 0$, hence $\Phi_{ii} = \Phi_{ii}^+$. From $\Phi_{ij} = \Phi_{ji}$ it follows that the Shapley value of player i can be written as the sum of three components:

$$\Phi_i = \Phi_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \Phi_{ji}^+ - \sum_{\substack{j=1 \\ j \neq i}}^{n} \Phi_{ji}^-. \tag{3.3}$$

These three components are the weighted fair share Φ_{ii} of player i, plus the voting power he can exert conditional on the presence of other players, minus the voting power he is prevented from exerting due to the presence of other players.

Let us analyze how the values of the members of the European Union changed in the European Council of Ministers after the enlargements of the Union in 1973, 1981, 1986, and 1995, respectively. Table 3.1 shows the members, the weights w_i, the Shapley values $\Phi_i(v)$, and the quorum for a qualified majority vote.

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Table 3.1. Votes and Shapley values in the European Council of Ministers 1981–1995

	1981		1985		1986		1995	
Member	w_i	Φ_i%	w_i	Φ_i%	w_i	Φ_i%	w_i	Φ_i%
D	10	17.86	10	17.38	10	13.42	10	11.67
F	10	17.86	10	17.38	10	13.42	10	11.67
I	10	17.86	10	17.38	10	13.42	10	11.67
GB	10	17.86	10	17.38	10	13.42	10	11.67
E	–	–	–		8	11.13	8	9.55
B	5	8.10	5	7.14	5	6.37	5	5.52
G	–	–	5	7.14	5	6.37	5	5.52
P	–	–	–		5	6.37	5	5.52
NL	5	8.10	5	7.14	5	6.37	5	5.52
DK	3	5.71	3	3.02	3	4.26	3	3.53
IRL	3	5.71	3	3.02	3	4.26	3	3.53
LUX	2	0.95	2	3.02	2	1.18	2	2.07
AU	–	–	–		–	–	4	4.54
S	–	–	–		–	–	4	4.54
FI	–	–	–		–	–	3	3.53
SUM	58	100	63	100	76	100	87	100
Quorum	41		45		54		62	

Tables 3.2–3.5 show the value of row member i with 10, 5, 3, or 2 votes to column member j with 10, 5, 3, or 2 votes. The diagonal values represent the value of a player with w votes to *another* player with the same number of votes, where the value Φ_{ii} of player i to himself is given in the second column from the right.

The tables show that the Shapley value for players with 10 and 5 votes declined from 1981 to 1995. The largest players thus lost voting power. Observe that the relative importance among the smaller countries with 3 or 2 votes was changing during the period. Between 1981 and 1985, the two countries with three votes – Denmark and Ireland – blocked those with five votes in the sense that the presence of the former prevented the latter from exercising voting power, as indicated by the negative entry (−0.0055) in the $[\Phi_{ij}]$-matrix in Table 3.2. Since the matrix is symmetric, the 5-vote countries

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Table 3.2. Power matrix of the European Council of Ministers 1981

1981, Φ-matrix						
	10	**5**	**3**	**2**	Φ_{ii}	Φ_i
10	0.0204	0.0119	0.0089	0.0029	0.0729	0.1786
5	0.0119	0.0048	−0.0055	0.0032	0.0366	0.0810
3	0.0089	−0.0055	0.0136	−0.0071	0.0262	0.0571
2	0.0029	0.0032	−0.0071	–	0.0055	0.0095
Positive power matrix						
	10	**5**	**3**	**2**	Φ_{ii}^+	Φ_i^+
10	0.0567	0.0277	0.0200	0.0044	0.0729	0.3427
5	0.0277	0.0250	0.0147	0.0043	0.0366	0.2061
3	0.0200	0.0147	0.0207	0.0000	0.0262	0.1561
2	0.0044	0.0043	0.0000	–	0.0055	0.0319
Negative power matrix						
	10	**5**	**3**	**2**	Φ_{ii}^-	Φ_i^-
10	0.0363	0.0159	0.0111	0.0015	0.0000	0.1641
5	0.0159	0.0202	0.0202	0.0011	0.0000	0.1251
3	0.0111	0.0202	0.0071	0.0071	0.0000	0.0989
2	0.0015	0.0011	0.0071	–	0.0000	0.0224

likewise blocked the 3-vote countries. Furthermore, the two 5-vote countries Belgium and the Netherlands were of positive value to each other. The additional 5-vote country Greece, entering in 1985, partly changed this picture. The value of Denmark and Ireland to the Netherlands, Belgium, and Greece, turned positive, whereas the importance of the 5-vote countries to each other turned negative. The reason for this was that the blocking capacity among the 5-vote countries increased by 0.0026 whereas the positive power for each country within this group declined from 0.0250 to 0.0195.

Another interesting feature of the $[\Phi_{ij}]$ matrix is the ranking of the players by their importance to another player, which differs from the ranking by weights. E.g., consider Table 3.4. Luxemburg with two votes was more important for 10-vote countries than member states with 3 votes, which indicates that Luxemburg contributed more to the voting power of 10-vote countries than Ireland or Denmark. Likewise, in 1995, the importance of the new 4-vote countries to Spain (8 votes) surmounted the importance of the five-vote

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Table 3.3. Power matrix of the European Council of Ministers 1985

1985, Φ-matrix						
	10	**5**	**3**	**2**	Φ_{ii}	Φ_i
10	0.0181	0.0111	0.0057	0.0057	0.0692	0.1738
5	0.0111	−0.0033	0.0005	0.0005	0.0323	0.0714
3	0.0057	0.0005	−0.0044	−0.0044	0.0147	0.0302
2	0.0057	0.0005	−0.0044	–	0.0147	0.0302
Positive power matrix						
	10	**5**	**3**	**2**	Φ_{ii}^+	Φ_i^+
10	0.0539	0.0247	0.0113	0.0113	0.0692	0.3392
5	0.0247	0.0195	0.0091	0.0091	0.0323	0.1975
3	0.0113	0.0091	0.0074	0.0074	0.0147	0.1019
2	0.0113	0.0091	0.0074	–	0.0147	0.1019
Negative power matrix						
	10	**5**	**3**	**2**	Φ_{ii}^-	Φ_i^-
10	0.0358	0.0137	0.0056	0.0056	0.0000	0.1653
5	0.0137	0.0228	0.0086	0.0086	0.0000	0.1261
3	0.0056	0.0086	0.0118	0.0118	0.0000	0.0718
2	0.0056	0.0086	0.0118	–	0.0000	0.0718

countries by 0.0013, and Luxemburg was almost as important for Spain as the five-vote countries. Hence, although the Shapley value Φ_i is always larger than Φ_j if $w_i > w_j$, the value Φ_{ik} of player i to player k may be smaller than Φ_{jk}, even when $w_i > w_j$.

It is interesting to consider which countries prefer which other countries to be present or not present in the EU. As for enlarging the EU beyond 15 members, the Amsterdam IGC (see doc. CONF/3815/97) hypothesizes the votes Poland (8), Romania (6), Czech Republic (5), Hungary (5), Bulgaria (4), Slovakia (3), Lithuania (3), Latvia (3), Slovenia (3), Estonia (3), Cypris (2), Malta (2) in a linear extrapolation to 27 member states.[5] We propose the following procedure for enlargement. Table 3.6 shows the Φ-Matrix for the

[5] Norway would get 3 votes.

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Table 3.4. Power matrix of the European Council of Ministers 1986

1986, Φ-matrix							
	10	**8**	**5**	**3**	**2**	Φ_{ii}	Φ_i
10	0.0144	0.0105	0.0053	0.0022	0.0031	0.0520	0.1342
8	0.0105	–	0.0029	0.0085	−0.0031	0.0435	0.1113
5	0.0053	0.0029	0.0031	0.0016	0.0017	0.0256	0.0637
3	0.0022	0.0085	0.0016	0.0068	−0.0049	0.0173	0.0426
2	0.0031	−0.0031	0.0017	−0.0049	–	0.0056	0.0118
Positive power matrix							
	10	**8**	**5**	**3**	**2**	Φ_{ii}^+	Φ_i^+
10	0.0407	0.0334	0.0193	0.0126	0.0047	0.0520	0.3150
8	0.0334	–	0.0183	0.0142	0.0025	0.0435	0.2815
5	0.0193	0.0183	0.0174	0.0115	0.0040	0.0256	0.2005
3	0.0126	0.0142	0.0115	0.0132	0.0016	0.0173	0.1430
2	0.0047	0.0025	0.0040	0.0016	–	0.0056	0.0462
Negative power matrix							
	10	**8**	**5**	**3**	**2**	Φ_{ii}^-	Φ_i^-
10	0.0264	0.0229	0.0140	0.0105	0.0016	0.0000	0.1808
8	0.0229	–	0.0154	0.0057	0.0057	0.0000	0.1702
5	0.0140	0.0154	0.0143	0.0100	0.0023	0.0000	0.1367
3	0.0105	0.0057	0.0100	0.0065	0.0065	0.0000	0.1004
2	0.0016	0.0057	0.0023	0.0065	–	0.0000	0.0343

adoption of one 16th country carrying 8, 6, 5, 4, 3, 2 votes respectively, where the quorum is determined so that the minimum percentage of votes for qualified majority is as close to the range from 70.69% to 71.43% (evidenced from Table 3.1) as possible.

Table 3.6 shows 1. The Shapley value of Luxemburg increases through the adoption of a 2-vote or 3-vote country. 2. The Shapley values of large countries decline through adoption. 3. The value of the largest countries to each other decline through the adoption of additional countries. Table 3.6 is equivalently set up for the adoption of an arbitrary number of countries carrying an arbitrary number of votes. This allows each country to assess the

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Table 3.5. Power matrix of the European Council of Ministers 1995

1995, Φ-matrix								
	10	**8**	**5**	**4**	**3**	**2**	Φ_{ii}	Φ_i
10	0.0111	0.0074	0.0046	0.0031	0.0018	0.0012	0.0447	0.1167
8	0.0074	–	0.0028	0.0041	0.0024	0.0026	0.0367	0.0955
5	0.0046	0.0028	0.0020	0.0013	0.0008	0.0007	0.0221	0.0552
4	0.0031	0.0041	0.0013	−0.0010	0.0024	−0.0006	0.0181	0.0454
3	0.0018	0.0024	0.0008	0.0024	0.0013	0.0012	0.0139	0.0353
2	0.0012	0.0026	0.0007	−0.0006	0.0012	–	0.0083	0.0207
Positive power matrix								
	10	**8**	**5**	**4**	**3**	**2**	Φ_{ii}^+	Φ_i^+
10	0.0349	0.0281	0.0167	0.0135	0.0102	0.0062	0.0447	0.3083
8	0.0281	–	0.0159	0.0135	0.0101	0.0063	0.0367	0.2763
5	0.0167	0.0159	0.0151	0.0123	0.0094	0.0057	0.0221	0.2083
4	0.0135	0.0135	0.0123	0.0113	0.0098	0.0052	0.0181	0.1805
3	0.0102	0.0101	0.0094	0.0098	0.0093	0.0056	0.0139	0.1463
2	0.0062	0.0063	0.0057	0.0052	0.0056	–	0.0083	0.0890
Negative power matrix								
	10	**8**	**5**	**4**	**3**	**2**	Φ_{ii}^-	Φ_i^-
10	0.0238	0.0207	0.0121	0.0104	0.0084	0.0050	0.0000	0.1916
8	0.0207	–	0.0131	0.0094	0.0077	0.0037	0.0000	0.1808
5	0.0121	0.0131	0.0130	0.0109	0.0086	0.0050	0.0000	0.1532
4	0.0104	0.0094	0.0109	0.0123	0.0074	0.0058	0.0000	0.1352
3	0.0084	0.0077	0.0086	0.0074	0.0080	0.0044	0.0000	0.1110
2	0.0050	0.0037	0.0050	0.0058	0.0044	–	0.0000	0.0684

value of each other country with a given number of votes in every conceivable adoption scenario. It is too space-consuming to set up all scenarios in this article. The challenge for each country is to assess which scenarios are most realistic, set up the analog of Table 3.6 for that scenario, and produce policy recommendations for alternative scenarios of adoption evaluated to be more beneficial.

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Table 3.6. *Φ*-Matrix with one adopted 16th country carrying 8, 6, 5, 4, 3, 2 votes respectively

Φ-Matrix with adopted 16th 8-vote country, sum = 95, quorum = 68; 71.58% majority								
	10	**8**	**5**	**4**	**3**	**2**	Φ_{ii}	Φ_i
10	0.0087	0.0064	0.0038	0.0032	0.0023	0.0020	0.0406	0.1100
8	0.0064	0.0053	0.0030	0.0025	0.0017	0.0004	0.0323	0.0856
5	0.0038	0.0030	0.0015	0.0013	0.0009	0.0007	0.0199	0.0515
4	0.0032	0.0025	0.0013	−0.0027	0.0006	−0.0001	0.0145	0.0366
3	0.0023	0.0017	0.0009	0.0006	0.0003	0.0000	0.0119	0.0302
2	0.0020	0.0004	0.0007	−0.0001	0.0000	–	0.0073	0.0192
Φ-Matrix with adopted 16th 6-vote country, sum = 93, quorum = 66; 70.97% majority								
10	0.0092	0.0065	0.0039	0.0033	0.0022	0.0021	0.0424	0.1122
8	0.0065	–	0.0030	0.0028	0.0020	0.0003	0.0337	0.0874
5	0.0039	0.0030	0.0017	0.0014	0.0009	0.0008	0.0208	0.0527
4	0.0033	0.0028	0.0014	−0.0009	0.0009	0.0002	0.0159	0.0394
3	0.0022	0.0020	0.0009	0.0009	0.0006	−0.0006	0.0122	0.0304
2	0.0021	0.0003	0.0008	0.0002	−0.0006	–	0.0080	0.0192
Φ-Matrix with adopted 16th 5-vote country, sum = 92, quorum = 65; 70.65% majority								
10	0.0094	0.0068	0.0039	0.0032	0.0023	0.0024	0.0436	0.1138
8	0.0068	–	0.0030	0.0032	0.0018	0.0001	0.0347	0.0889
5	0.0039	0.0030	0.0017	0.0014	0.0010	0.0009	0.0214	0.0534
4	0.0032	0.0032	0.0014	−0.0035	0.0013	−0.0073	0.0156	0.0381
3	0.0023	0.0018	0.0010	0.0013	0.0003	−0.0006	0.0127	0.0312
2	0.0024	0.0001	0.0009	−0.0073	−0.0006	–	0.0080	0.0188

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Table 3.6. (continued)

Φ-Matrix with adopted 16th 4-vote country, sum = 91, quorum = 65; 71.43% majority								
10	0.0100	0.0072	0.0042	0.0030	0.0022	0.0019	0.0426	0.1142
8	0.0072	–	0.0030	0.0019	0.0027	0.0004	0.0336	0.0884
5	0.0042	0.0030	0.0017	0.0013	0.0010	0.0008	0.0210	0.0537
4	0.0030	0.0019	0.0013	−0.0025	0.0004	0.0011	0.0167	0.0428
3	0.0022	0.0027	0.0010	0.0004	0.0011	−0.0007	0.0121	0.0305
2	0.0019	0.0004	0.0008	0.0011	−0.0007	–	0.0082	0.0202
Φ-Matrix with adopted 16th 3-vote country, sum = 90, quorum = 64; 71.11% majority								
10	0.0100	0.0072	0.0042	0.0032	0.0024	0.0017	0.0436	0.1152
8	0.0072	–	0.0031	0.0019	0.0021	0.0008	0.0342	0.0884
5	0.0042	0.0031	0.0017	0.0014	0.0011	0.0006	0.0214	0.0541
4	0.0032	0.0019	0.0014	0.0051	0.0003	0.0016	0.0180	0.0457
3	0.0024	0.0021	0.0011	0.0003	0.0006	0.0001	0.0122	0.0303
2	0.0017	0.0008	0.0006	0.0016	0.0001	–	0.0088	0.0217
Φ-Matrix with adopted 16th 2-vote country, sum = 89, quorum = 63; 70.79% majority								
10	0.0103	0.0074	0.0042	0.0033	0.0024	0.0015	0.0447	0.1166
8	0.0074	–	0.0032	0.0013	0.0028	0.0008	0.0353	0.0902
5	0.0042	0.0032	0.0019	0.0014	0.0011	0.0006	0.0219	0.0548
4	0.0033	0.0013	0.0014	0.0036	−0.0002	0.0018	0.0176	0.0439
3	0.0024	0.0028	0.0011	−0.0002	0.0013	−0.0001	0.0128	0.0313
2	0.0015	0.0008	0.0006	0.0018	−0.0001	0.0002	0.0087	0.0214

Appendix 1:
Decomposition of the Shapley value: The *Φ*-Matrix

Define $\Phi_j(S)$ as j's Shapley value given the set S of all players and the respective characteristic function $v(R)$ defined over all $R \subseteq S$. We define the value of player i to player j as

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$$\Phi_{ij} = \sum_{\substack{S \subseteq N \\ i,j \in S}} \frac{(n-s)!(s-1)!}{n!} (\Phi_j(S) - \Phi_j(S \backslash \{i\})). \tag{A1}$$

Denote the marginal contribution of player j to $v(S)$ as $M_j(S) = v(S) - v(S \backslash \{j\})$ and $\lambda(N, S) = \frac{(s-1)!(n-s)!}{n!}$ as the Shapley factor. Hence

$$\Phi_{ij} = \sum_{\substack{S \subseteq N \\ i,j \in S}} \lambda(N,S) \times \left(\sum_{R \subseteq S} \lambda(S,R) \cdot M_j(R) - \sum_{R \subseteq S \backslash \{i\}} \lambda(S \backslash \{i\}, R) \cdot M_j(R) \right) \tag{A2}$$

$$= \sum_{\substack{S \subseteq N \\ i,j \in S}} \lambda(N,S) \sum_{R \subseteq S} \lambda(S,R) \cdot M_j(R) - \sum_{\substack{S \subseteq N \\ i,j \in S}} \lambda(N,S) \sum_{R \subseteq S \backslash \{i\}} \lambda(S \backslash \{i\}, R) \cdot M_j(R)$$

$$= \sum_{\substack{S \subseteq N \\ i,j \in S}} \lambda(N,S) \sum_{\substack{R \subseteq S \\ i \in R}} \lambda(S,R) \cdot M_j(R) + \sum_{\substack{S \subseteq N \\ i,j \in S}} \lambda(N,S) \sum_{\substack{R \subseteq S \\ i \notin R}} \lambda(S,R) \cdot M_j(R) - \sum_{\substack{S \subseteq N \\ i,j \in S}} \lambda(N,S) \sum_{R \subseteq S \backslash \{i\}} \lambda(S \backslash \{i\}, R) \cdot M_j(R). \tag{A3}$$

Observe the two stages in the selection of the coalitions R. First, S is selected from all subsets of N including both i and j. From every S, the subsets R are taken, but the selection of the R in the first sum differs from the selections in the second and third sums. In the first sum the subsets R include all subsets of S which have i as a member and only such subsets. In the second and third sums only such subsets are chosen from every S which do not have i as a member. The two stage selection of $R \subseteq S$ implies that every set R is accounted for more than once. There are $\sum_{s=r}^{n} \binom{n-r}{s-r}$ possibilities for a coalition of size r to appear in the first sum, depending on the respective coalition $S \subseteq N$ selected in the first stage. Respectively, we have $\sum_{s=r+1}^{n} \binom{n-r-1}{s-r-1}$ possibilities for a coalition of size r to appear in the second or the third sum. Hence

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$$\Phi_{ij} = \sum_{\substack{R\subseteq N\\ i\in R}} \sum_{s=r}^{n} \frac{(n-s)!(s-1)!(r-1)!(s-r)!}{n!s!} \binom{n-r}{s-r} M_j(R)$$

$$+ \sum_{\substack{R\subseteq N\\ i\notin R}} \sum_{s=r+1}^{n} \frac{(n-s)!(s-1)!(r-1)!(s-r)!}{n!s!} \binom{n-r-1}{s-r-1} M_j(R) \quad \text{(A4)}$$

$$- \sum_{\substack{R\subseteq N\\ i\notin R}} \sum_{s=r+1}^{n} \frac{(n-s)!(s-1)!(r-1)!(s-1-r)!}{n!(s-1)!} \binom{n-r-1}{s-r-1} M_j(R)$$

$$= \sum_{\substack{R\subseteq N\\ i\in R}} \frac{(r-1)!(n-r)!}{n!} M_j(R) \sum_{s=r}^{n} \frac{1}{s}$$

$$+ \sum_{\substack{R\subseteq N\\ i\notin R}} \sum_{s=r+1}^{n} \frac{(r-1)!(s-r)(n-r-1)!}{n!s} M_j(R) \quad \text{(A5)}$$

$$- \sum_{\substack{R\subseteq N\\ i\notin R}} \sum_{s=r+1}^{n} \frac{(r-1)!(n-r-1)!}{n!} M_j(R)$$

$$= \sum_{\substack{R\subseteq N\\ i\in R}} \frac{(r-1)!(n-r)!}{n!} M_j(R) \sum_{s=r}^{n} \frac{1}{s}$$

$$+ \sum_{\substack{R\subseteq N\\ i\notin R}} \frac{(r-1)!(n-r-1)!}{n!} M_j(R) \sum_{s=r+1}^{n} \frac{s-r}{s} \quad \text{(A6)}$$

$$- \sum_{\substack{R\subseteq N\\ i\notin R}} \frac{(r-1)!(n-r-1)!}{n!} M_j(R)(n-r)$$

$$= \sum_{\substack{R\subseteq N\\ i\in R}} \frac{(r-1)!(n-r)!}{n!} M_j(R) \sum_{s=r}^{n} \frac{1}{s}$$

$$+ \sum_{\substack{R\subseteq N\\ i\notin R}} \frac{(r-1)!(n-r-1)!}{(n-1)!} M_j(R) \frac{1}{n} \left(\sum_{s=r+1}^{n} \left(\frac{s-r}{s} \right) - (n-r) \right) \quad \text{(A7)}$$

$$= \sum_{\substack{R\subseteq N\\ i\in R}} \frac{(r-1)!(n-r)!}{n!} M_j(R) \sum_{s=r}^{n} \frac{1}{s}$$

$$+ \sum_{\substack{R\subseteq N\\ i\notin R}} \frac{(r-1)!(n-r-1)!}{(n-1)!} M_j(R) \frac{1}{n} \sum_{s=r+1}^{n} \left(\frac{s-r}{s} - \frac{s}{s} \right) \quad \text{(A8)}$$

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$$= \sum_{\substack{R \subseteq N \\ i \in R}} \frac{(r-1)!(n-r)!}{n!} M_j(R) \sum_{s=r}^{n} \frac{1}{s}$$

$$- \sum_{\substack{R \subseteq N \\ i \notin R}} \frac{(r-1)!(n-r-1)!}{(n-1)!} M_j(R) \frac{r}{n} \sum_{s=r+1}^{n} \frac{1}{s} \tag{A9}$$

$$= \sum_{R \subseteq N} \frac{(r-1)!(n-r)!}{n!} (M_j(R) - M_j(R\backslash\{i\})) \sum_{s=r}^{n} \frac{1}{s}. \tag{A10}$$

This is equivalent to

$$\Phi_{ij} = \sum_{R \subseteq N} \frac{(r-1)!(n-r)!}{n!} ((v(R) - v(R\backslash\{j\}))$$

$$- (v(R\backslash\{i\}) - v(R\backslash\{i,j\}))) \sum_{s=r}^{n} \frac{1}{s}. \tag{A11}$$

Appendix 2:
The marginal contribution of a player: The Ψ-Matrix

Ψ_{ij} is defined as follows:

$$\Psi_{ij} = \sum_{R \subseteq N} \frac{(r-1)!(n-r)!}{n!} ((v(R) - v(R\backslash\{j\})) - (v(R\backslash\{i\}) - v(R\backslash\{i,j\})))$$

$$= \sum_{R \subseteq N} \frac{(r-1)!(n-r)!}{n!} (M_j(R) - M_j(R\backslash\{i\})). \tag{A12}$$

In Ψ_{ij} the term $(M_j(R) - M_j(R\backslash\{i\}))$ is not weighted by $\sum_{s=r}^{n} \frac{1}{s}$ which is the only difference to Φ_{ij}. Observe that the $[\Phi_{ij}]$-matrix is constructed by accounting for i's marginal contributions to player j's fair share or voting power in *all* possible subgames $\{v(R), R \subseteq S\}$, $S \subseteq N$, whereas $[\Psi_{ij}]$ accounts only for the case $S = N$. We now show that $\Psi_{ij} = \Phi_j(N) - \Phi_j(N\backslash\{i\})$.

$$\Psi_{ij} = \sum_{R \subseteq N} \frac{(r-1)!(n-r)!}{n!}$$

$$\times ((v(R) - v(R\backslash\{j\})) - (v(R\backslash\{i\}) - v(R\backslash\{i,j\}))) \tag{A13}$$

$$= \sum_{\substack{R \subseteq N \\ R \ni i,j}} \lambda(N,R) M_j(R) - \sum_{\substack{R \subseteq N \\ R \ni i,j}} \lambda(N,R) M_j(R\backslash\{i\})$$

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$$= \sum_{\substack{R \subseteq N \\ R \ni i,j}} \lambda(N,R)M_j(R) - \sum_{\substack{R \subseteq N\setminus\{i\} \\ R \ni j}} \lambda(N\setminus\{i\},R)\frac{r}{n}M_j(R)$$

$$= \sum_{\substack{R \subseteq N \\ R \ni j}} \lambda(N,R)M_j(R) - \sum_{\substack{R \subseteq N\setminus\{i\} \\ R \ni j}} \lambda(N\setminus\{i\},R)\frac{n-r}{n}M_j(R)$$

$$- \sum_{\substack{R \subseteq N\setminus\{i\} \\ R \ni j}} \lambda(N\setminus\{i\},R)\frac{r}{n}M_j(R)$$

$$= \sum_{\substack{R \subseteq N \\ R \ni j}} \lambda(N,R)M_j(R)$$

$$- \left(\sum_{\substack{R \subseteq N\setminus\{i\} \\ R \ni j}} \lambda(N\setminus\{i\},R)\frac{n-r}{n}M_j(R) + \sum_{\substack{R \subseteq N\setminus\{i\} \\ R \ni j}} \lambda(N\setminus\{i\},R)\frac{r}{n}M_j(R) \right)$$

$$= \sum_{\substack{R \subseteq N \\ R \ni i,j}} \lambda(N,R)M_j(R) - \sum_{\substack{R \subseteq N\setminus\{i\} \\ R \ni j}} \lambda(N\setminus\{i\},R)M_j(R). \qquad \text{(A14)}$$

Hence

$$\Psi_{ij} = \Phi_j(N) - \Phi_j(N\setminus\{i\}).$$

Ψ_{ij} is the difference between the Shapley value $\Phi_j(N)$ of player j in the game played by all players in N and the Shapley value $\Phi_j(N\setminus\{i\})$ of player j in the game played by all players except i. Hence Ψ_{ij} may be interpreted as player i's *marginal contribution* to player j's fair share in the game $\{v(S), S \subseteq N\}$. In simple games this may be referred to as player i's *marginal contribution* to player j's *voting-power*. From efficiency of Φ_j it follows that $\Psi_i = \sum_{j=1}^{n} \Psi_{ij} = v(N) - v(N\setminus\{i\})$:

$$\sum_{j=1}^{n} \Psi_{ij} = \sum_{j=1}^{n} (\Phi_j(N) - \Phi_j(N\setminus\{i\}))$$

$$= v(N) - \sum_{j=1}^{n} \Phi_j(N\setminus\{i\})$$

$$= v(N) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \Phi_j(N\setminus\{i\}) - \Phi_i(N\setminus\{i\})$$

$$= v(N) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \Phi_j(N\setminus\{i\}) \qquad \text{(A15)}$$

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Hence,

$$\Psi_i = v(N) - v(N \backslash \{i\}).$$

It follows that

$$\sum_{i=1}^{n} \Psi_i = n \cdot v(N) - \sum_{i=1}^{n} v(N \backslash \{i\}). \quad \text{(A16)}$$

Hence the sum of the Ψ_{ij}'s is the marginal contribution of player i to the value of the grand coalition N. Obviously, the matrix $[\Psi_{ij}]$ is symmetric. Furthermore, observe that for the diagonal elements $\Psi_{ii} = \Phi_i(N)$. The marginal contribution of a player to himself is equal to his Shapley value.

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