FROM INSIGHTS INTO GAME THEORY BY EW-YA GURA AND M.B. MASCHLER

The Axioms:

1. The function f assigns a preference order, called a *social preference*, to every preference profile.
2. The function f reflects a positive association between the preferences of the individuals in society and the social preference: if in a certain preference profile f establishes the preference order $\begin{smallmatrix}x\\y\end{smallmatrix}$ and the preference profile changes so that more individuals in society prefer x to y, or $\begin{smallmatrix}y\\x\end{smallmatrix}$ becomes y~x, then f establishes that x is preferred to y in the new preference profile too.
3. The function f obeys a unanimous choice: if all the individuals in society prefer x to y, f also establishes that x is preferred to y.
4. The function f establishes a preference between alternatives x and y independently of any other alternatives; that is, what f establishes depends on the preference orders of the individuals in society with regard only to x and y.

Examples:

1. Given that f satisfies the above axioms, what will f establish as the social preference if the preference profile of a society of three individuals is:

1	**2**	**3**
x	z	x
y	x	z
z	y	y
t	t	t

Solution:

The function f will establish the preference order $\begin{smallmatrix}x\\y\end{smallmatrix}$ because everybody prefers it (unanimous decision axiom). For the same reason, f will establish the preference relations $\begin{smallmatrix}z\\t\end{smallmatrix}$ $\begin{smallmatrix}y\\t\end{smallmatrix}$ $\begin{smallmatrix}x\\t\end{smallmatrix}$; hence, by the unanimous decision axiom, f will establish the preference orders $\begin{smallmatrix}x\\y\end{smallmatrix}$ $\begin{smallmatrix}z\\t\end{smallmatrix}$ $\begin{smallmatrix}y\\t\end{smallmatrix}$ $\begin{smallmatrix}x\\t\end{smallmatrix}$.

The independence of irrelevant alternatives axiom applies here as well, because the preferences described above were established independently of the preferences of the individuals in society with regard to the other alternatives.

For example, $\begin{smallmatrix}x\\y\end{smallmatrix}$ is independent of the preference orders with regard to z and t; $\begin{smallmatrix}z\\t\end{smallmatrix}$ is independent of the preference orders with regard to x and y, and so on.

We still do not know what f will establish either with regard to alternatives x and z or with regard to alternatives y and z, because as far as these alternatives are concerned society is divided in its opinions.

If we knew more properties of f, we could perhaps say more. For example, if we assumed that f establishes the preference order $\begin{smallmatrix}y\\z\end{smallmatrix}$, we could conclude, by the transitivity property of the preference relation, that f also establishes the preference order $\begin{smallmatrix}x\\z\end{smallmatrix}$. That is, we would have that $\begin{smallmatrix}x\\y\end{smallmatrix}$ $\begin{smallmatrix}y\\z\end{smallmatrix}$ $\begin{smallmatrix}z\\t\end{smallmatrix}$ $\begin{smallmatrix}y\\t\end{smallmatrix}$ $\begin{smallmatrix}x\\t\end{smallmatrix}$, and therefore we could say that:

$$f\begin{pmatrix} x & z & x \\ y & x & z \\ z & y & y \\ t & t & t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

and thus establish f for this example.

2. Given that f satisfies the above axioms:

I. What will f establish if the preference profile in a society of three individuals is:

1	**2**	**3**
x	z	x
z	x	y
y	y	z

Solution:

By the unanimous decision axiom, we can say that f will establish as the social preference that x is preferred to y, because all individuals in society prefer it. Moreover, the independence of irrelevant alternatives axiom indirectly applies here, because this preference was established independently of society's preferences with regard to z. The given preference profile does not enable us to say anything about the social preference with regard to the pairs x, z or y, z, because society is divided in its opinions with regard to these alternatives: some individuals prefer one possibility and others prefer the other.

II. Given the same preference profile, what can you say about f if it is known that f establishes that y is preferred to z?

Solution:

We know that: $\begin{matrix} x & y \\ y & z \end{matrix}$. By the transitivity of the preference relation, f must also establish that x is preferred to z.

$$f\begin{pmatrix} x & z & x \\ z & x & y \\ y & y & z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This example is of interest because it illustrates a *prediction*. Although we did not know in advance what f would decide with regard to alternatives x and z, we have managed to predict the social decision.

Remark:

If additional information were to reveal that, say, f establishes a preference for z over y, we would not be able to predict the social decision. In this case the information would amount to $\begin{matrix} x & z \\ y & y \end{matrix}$, from which it is impossible to conclude how f would order the pair of alternatives x, z.

3. Given that f satisfies all four axioms, describe what the social decision will be given the following preference profile, if it is also known that f establishes a preference for y over z.

1	**2**	**3**
x	z	x
y	x	z
z	y	y
t	t	t

Solution:

In many cases it is possible to know what f establishes, like when all individuals in society prefer a certain alternative to another alternative (unanimous decision). Let us compile all the available information:

$$\begin{matrix} x & x & y & z & y \\ y & t & t & t & z \end{matrix}$$

The first four columns of preferences from the left were obtained by the unanimous decision axiom. The last column is additional information. According to the available information and by the transitivity of the preference relation, we get $\begin{matrix} x \\ z \end{matrix}$. Now we have all the information and we may conclude that:

$$f\begin{pmatrix} x & z & x \\ y & x & z \\ z & y & y \\ t & t & t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

2.8 EXERCISES

1. (1) Describe the social decision given the following preference profile:

1	**2**
x	z
t	x
y	t
z	y

(2) If we know that f establishes a preference for x over z, what will the social decision be?

(3) What other information must we have in order to know what the social decision will be?

2. (1) Can we predict the social decision given the following preference profile?

1	2	3
x	z	y
y	x	x
z	y	z

(2) If we know that the social decision establishes a preference for x over y, what will the social choice function be?

(3) Can we predict the social decision given the following preference profile if, besides the information in (2), we also know that f establishes $\genfrac{}{}{0pt}{}{y}{z}$?

1	2	3
x	x	y
y	z	x
z	y	z

3. (1) Can we predict the social decision given the following preference profile?

1	2	3
x	x	x
z	z	z
y	y	t
t	t	y

If so, state the social decision; if not, provide the missing information.

(2) Describe the social decision given the preference profile above, when it is known that f establishes a preference for t over y.

(3) Describe the social decision given the preference profile above, when it is known that f establishes a preference for y over t.

2.9 ARROW'S THEOREM

The present chapter opened with a discussion of the shortcomings of majority rule as a principal method of social decision-making. The question that was raised in light of these shortcomings was whether it is possible to find another, "fair" way of making a social decision. In the course of this chapter we constructed a system of axioms, that is, a system of intuitive requirements for a fair decision-making procedure. The question now is whether there is a social decision rule for all possible preference profiles that satisfies this system of axioms.

Kenneth Arrow's surprising answer is that there is no social choice function that satisfies all the axioms! This means that every social choice function we can think of will fail to satisfy at least one of the axioms. In other words, there is an internal contradiction in the system of axioms presented in this chapter.

In this section we shall prove Arrow's theorem about the nonexistence of a social choice function that satisfies all the axioms. In the course of the proof we shall see that any decision rule that does satisfy Axioms 1–4 must be a dictatorial decision rule, which contradicts Axiom 5, defined in Section 2.5 (p. 79).

For the proof, we assume that a decision rule satisfies Axioms 1–4.

Definition:

A set of individuals V is said to be *decisive for the pair* $\genfrac{}{}{0pt}{}{x}{y}$ if, *for every preference profile* in which everyone in V prefers $\genfrac{}{}{0pt}{}{x}{y}$ and everyone else prefers $\genfrac{}{}{0pt}{}{y}{x}$, the social choice function f establishes $\genfrac{}{}{0pt}{}{x}{y}$.

In other words, V is decisive for $\genfrac{}{}{0pt}{}{x}{y}$ if f establishes $\genfrac{}{}{0pt}{}{x}{y}$ whenever the members of V have this preference and the other members have the opposite preference.

Note: If there is a dictator in the society, then he constitutes a decisive set for every pair of alternatives (and not for just one pair).

Remark: According to the definition, V is decisive for $\genfrac{}{}{0pt}{}{x}{y}$ if all members of V prefer $\genfrac{}{}{0pt}{}{x}{y}$ and all the other members prefer $\genfrac{}{}{0pt}{}{y}{x}$. But if some members not in V prefer $\genfrac{}{}{0pt}{}{x}{y}$ or x~y, then this profile favors $\genfrac{}{}{0pt}{}{x}{y}$ and by Axiom 2, the social choice function will continue to favor $\genfrac{}{}{0pt}{}{x}{y}$. In any case, the preference profile tends to favor x; therefore, by Axiom 2, the social choice function will decide in favor of $\genfrac{}{}{0pt}{}{x}{y}$.

Is There a Decisive Set?

The answer is yes! The set of *all individuals* is no doubt a decisive set, and not only for a certain pair $\genfrac{}{}{0pt}{}{x}{y}$, but for every pair, by the unanimous decision axiom (Axiom 3).

Consider the set of all individuals that, as we said, is decisive for every pair of alternatives. It might be possible for us to subtract some individuals from the set, so that the remaining set will still be decisive – if not for all pairs, then at least for one pair. We shall keep subtracting individuals from the set, as long as there remains a set of individuals that is decisive for some pair. We shall continue the procedure until we have a set that is still decisive for some pair $\genfrac{}{}{0pt}{}{x}{y}$, but from which no more individuals can be subtracted, because that would result in a set that is not decisive for any pair of alternatives. The smallest decisive set for any pair of alternatives is called a *minimal decisive* set.[2]

[2] This set cannot be the empty set, because if the empty set were decisive for the pair $\genfrac{}{}{0pt}{}{x}{y}$, the social choice function would establish $\genfrac{}{}{0pt}{}{x}{y}$ even if everyone preferred $\genfrac{}{}{0pt}{}{y}{x}$, which contradicts the unanimous decision axiom.

Definition:

Set V is called a *minimal decisive set* if V is decisive for a certain pair $\genfrac{}{}{0pt}{}{x}{y}$ and if subtracting individuals from the set results in a set that is not decisive for any pair of alternatives.

We have thus proved that there exists a minimal decisive set; that is, there exists a set that is decisive for a certain pair $\genfrac{}{}{0pt}{}{x}{y}$ and any strict subset of it is not decisive for any pair of alternatives.

Let V be a minimal decisive set. We denote the pair for which it is decisive by $\genfrac{}{}{0pt}{}{x}{y}$. Let j be a specific individual in V and let W be the set consisting of the remaining individuals in V. We denote the set of all individuals not in V by U. Now, consider the following preference profile:

$$\begin{array}{ccc} \multicolumn{2}{c}{\overbrace{\{j\}\quad W}^{V}} & U \\ \hline x & z & y \\ y & x & z \\ z & y & x \end{array}$$

For this preference profile the social choice function f will establish the preference relation $\genfrac{}{}{0pt}{}{x}{y}$, because all individuals in V = {j}∪W prefer x to y and V is a decisive set for the pair $\genfrac{}{}{0pt}{}{x}{y}$.

The social choice function will not be able to establish $\genfrac{}{}{0pt}{}{z}{y}$, because only the individuals in W prefer it and W is not a decisive set, because V is the minimal decisive set. We apply here the independence of irrelevant alternatives axiom (Axiom 4), which allows us to rule out the possibility that the position of x in the preference relations may affect the preference between y and z.

Hence, the social choice function will establish $\genfrac{}{}{0pt}{}{y}{z}$ or y~z.

Since we already have ${x \atop y}$, f will establish the preference relation ${x \atop z}$ by the transitivity of the preference relation. But only j prefers it, while everyone else prefers the contrary. Hence it follows by Axiom 4 that {j} constitutes a decisive set for the pair ${x \atop z}$. But V is a minimal decisive set and therefore $W = \emptyset$ and V={j}.

Moreover, because z denotes any alternative, it follows that {j} is decisive for every pair of alternatives ${x \atop z}$.

We have thus proved that if f satisfies Axioms 1–4, then there exists a minimal decisive set consisting of a single individual and it is decisive for the pairs ${x \atop z}$ for some x and any z. If no one remains $W = \emptyset$.

Our task now is to show that a decisive set of one individual is a dictator; that is, if V is a decisive set consisting of a single player, then this player must be decisive for every pair of alternatives and not only for pairs of the ${x \atop z}$ kind. As we said, {j} constitutes a decisive set for every pair of the ${x \atop z}$ kind.

Consider the following preference profile:

{j}	U
w	z
x	w
z	x

f will establish ${x \atop z}$, because {j} is decisive for these pairs.

f will also establish ${w \atop x}$, because everyone prefers it.

By the transitivity of the preference relation, f will establish ${w \atop z}$; that is, {j} is decisive for every pair of alternatives ${w \atop z}$, when $w \neq x$, $z \neq x$.

Finally, consider the following preference profile:

{j}	U
w	z
z	x
x	w

f will establish ${w \atop z}$ when $w \neq x$, $z \neq x$, because {j} is decisive for ${w \atop z}$. f will establish ${z \atop x}$, because everyone prefers it.

By the transitivity of the preference relation, f will establish ${w \atop x}$.

But only j prefers it, and so {j} is a decisive set for ${w \atop x}$.

Thus we have proved that:

{j} is decisive for ${x \atop z}$ for all z,

{j} is decisive for ${w \atop z}$ for all w and all z different from x, and

{j} is decisive for ${w \atop x}$ for all w.

Those are all the possibilities. Indeed, {j} is decisive for ${x \atop z}$ for all z, $z \neq x$, because z can be replaced by any alternative. Because {j} is decisive for ${w \atop z}$ for all w and all z different from x, w and z can be replaced by any alternative except x. As for the possibility $w = x$, we know that {j} is decisive for ${x \atop z}$. Also, {j} is decisive for ${w \atop x}$ for all w because w can be replaced by any alternative, and with that we have addressed all the possibilities.

We can therefore sum up by saying that {j} is decisive for every pair of alternatives and therefore j is a dictator!

To summarize, we started from the social choice function that satisfies Axioms 1–4 and proved that it is necessarily a dictatorial rule, which does not satisfy Axiom 5.

That is, there is no social choice function that satisfies the system of Axioms 1–5 in its entirety.

2.10 WHAT NEXT?

Our aim throughout this chapter has been to find a social decision rule that will satisfy our sense of fairness in a democratic society. This aim was not achieved; what is more, it was proved that such a rule does not exist!

What is to be done? What rule are we to follow in making decisions? How is society to conduct its affairs? From all that has been said in this chapter it follows that there are no satisfactory answers to these questions. We must accept the fact that every decision rule that is chosen will not satisfy at least one of Arrow's axioms.

Arrow's book, in which he proved the theorem that now bears his name, stirred debate among social scientists over the implications of the impossibility of a satisfactory decision rule. Social science philosophers suddenly realized that the question "What is good for society?" is not always possible to answer. Arrow's conclusion brought about a radical change in many scientists' perception of the human world around us.

Arrow's book also led to mathematical research. For example, mathematicians wondered whether they could avoid contradicting the axioms by restricting the domain of preference profiles. In fact, narrower domains were established in which all five axioms could be satisfied by appropriate social choice functions. There were also proposals to integrate lotteries into decision rules: if, for example, it were seen that majority rule leads to a cyclic preference relation, then

it would be decided by casting lots between the preferences. We shall not go into all that has been done in this area. We shall only note that Arrow's study spawned a huge literature, both theoretical and applied, on this fascinating subject.

2.11 REVIEW EXERCISES

1. Given the following preference profiles, what will the social choice function be when the guiding rule is to decide by a pairwise majority vote?

(1)

$$f\begin{pmatrix} x & t & y \\ y & x & t \\ z & z & x \\ t & y & z \end{pmatrix} =$$

(2)

$$f\begin{pmatrix} x\sim t & t & x\sim z \\ z & x & t \\ y & y & y \\ & z & \end{pmatrix} =$$

2. Given the following preference profiles, what will the social choice function be when society consists of two individuals and the guiding rule is: if both individuals prefer x to y, then society will prefer x to y; if they are divided in their opinions, then society will be indifferent to a choice between x and y; if both individuals are indifferent to a choice between x and y, then society will be indifferent to a choice between x and y; if one prefers x to y and the other is indifferent to a choice between them, then society will prefer x to y?

(1)

$$f\begin{pmatrix} x & y \\ y & z \\ z & x \end{pmatrix} =$$