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Probabilities of Pure Nash Equilibria in Matrix Games when the Payoff Entries of One Player Are Randomly Selected

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Abstract The Nash equilibrium in pure strategies represents an important solution concept in nonzero sum matrix games. Existence of Nash equilibria in games with known and with randomly selected payoff entries have been studied extensively. In many real games, however, a player may know his own payoff entries but not the payoff entries of the other player. In this paper, we consider nonzero sum matrix games where the payoff entries of one player are known, but the payoff entries of the other player are assumed to be randomly selected. We are interested in determining the probabilities of existence of pure Nash equilibria in such games. We characterize these probabilities by first determining the finite space of ordinal matrix games that corresponds to the infinite space of matrix games with random entries for only one player. We then partition this space into mutually exclusive spaces that correspond to games with no Nash equilibria and with r Nash equilibria. In order to effectively compute the sizes of these spaces, we introduce the concept of top-rated preferences minimal ordinal games. We then present a theorem which provides a mechanism for computing the number of games in each of these mutually exclusive spaces, which then can be used to determine the probabilities. Finally, we summarize the results by deriving the probabilities of existence of unique, nonunique, and no Nash equilibria, and we present an illustrative example.

Keywords Matrix games · Pure Nash equilibria · Ordinal games · Random payoffs

1 Introduction

Game theory has been researched extensively since it was first established in 1947 by Von Neumann and Morgenstern [1]. In 1956, Nash introduced the concept of an

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equilibrium solution, which became known as the Nash equilibrium for nonzero sum games [2]. Matrix games represent a fundamental special problem in game theory and are used to describe games where the strategy spaces are discrete. Strategies in matrix games can be either pure (deterministic) or mixed (probabilistic). It is well known that a matrix game may or may not have a pure Nash equilibrium (PNE), and if it does, the PNE may or may not be unique [2].

Consider a two-player nonzero-sum $n \times m$ matrix game where players P1 and P2 have strategies x_i and y_j which can take on values from the sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ respectively. Let the payoff entries for P1 and P2 corresponding to a strategy pair $\{x_i, y_j\}$ be $J_1(x_i, y_j)$ and $J_2(x_i, y_j)$ respectively. A Nash equilibrium for such a game is defined as a pair of strategies $\{x^N, y^N\} \in X \times Y$ which satisfies the following inequalities:

$$J_1(x^N, y^N) \le J_1(x_i, y^N), \quad \forall x_i \in X, \tag{1}$$

$$J_2(x^N, y^N) \le J_2(x^N, y_j), \quad \forall y_j \in Y. \tag{2}$$

If the payoff entries of both players are known, it is easy to determine the PNEs if they exist. If, on the other hand, the payoff entries are not known then one possible approach would be to assume that they are selected randomly from a continuous distribution over an interval. We refer to these types of games as random entries games or RE-games. A RE-game can have a unique PNE, nonunique PNEs, or no PNE. As a result, one would be interested in computing the probability of existence of PNEs when the payoffs are selected randomly. This problem has received considerable attention in the literature [3–11] especially for the case when the J_1 entries in each column are distinct and the J_2 entries in each row are also distinct. An expression for the probability that the game has exactly r PNEs, where r is an integer such that $r \le \mathrm{Min}(n, m)$, has been derived in [8]. It is also shown in [8] that as n and $m \to \infty$, this probability converges to $e^{-1}/r!$. This basically means that as n and $m \to \infty$ the probability of existence of a unique PNE converges to $e^{-1} = 0.36517$, the probability of existence of nonunique PNEs converges to $\sum_{r=2}^{\infty} \frac{e^{-1}}{r!} = 1 - 2e^{-1} = 0.26965$, and the probability of existence of no PNE converges to $e^{-1} = 0.36517$.

In this paper, we consider nonzero sum matrix games where the payoff entries of one player are known and fixed while the payoff entries of the other are unknown, selected randomly from a continuous distribution over an interval. We refer to these types of games as one player RE-games, or OPRE-games. OPRE-games are more realistic than general RE-games since in most cases, a player may typically know his payoff entries but may have no knowledge of the payoff entries of the other player, and therefore assumes that they are randomly selected. In this paper, we are interested in characterizing the probabilities that an OPRE-game has no PNE or r PNEs. We approach the derivation of these probabilities using concepts from ordinal games [12]. In Sect. 2, we discuss the relationship between a RE-game and its corresponding Minimal Ordinal game. In Sect. 3, we introduce the concept of top-rated preferences minimal ordinal games and in Sect. 4 we exploit this concept to determine the number of top-rated preferences minimal ordinal games of each type in OPRE-games. The probabilities of existence of no PNE and r PNEs are then easily determined. We summarize the results by deriving expression for the probabilities of existence of a unique PNE, nonunique PNEs, and no PNE and we present an illustrative example.

2 Minimal Ordinal Games

It is well known that any RE-game can be transformed into an equivalent ordinal game by replacing the payoff entries with a rank ordering of these entries. It is also well known that there is a one-to-one correspondence between the PNEs of a matrix game and the Nash equilibria of its equivalent ordinal game [12]. Clearly, while the number of RE-games of a given size is infinite, the number of ordinal matrix games of the same size is finite. Thus, instead of characterizing the PNEs within the infinite space of RE-games of a given size, we will exploit this property and characterize these equilibria in the equivalent, but finite, space of ordinal matrix games of the same size.

Following [12], a minimal ordinal game, or MO-game, is one where $J_1(x_i, y_j)$ are column ordered and $J_2(x_i, y_j)$ are row ordered. That is, for every $y_j \in Y$, the entries $J_1(x_i, y_j)$ are replaced by a rank ordering from 1 to n, and for every $x_i \in X$, the entries $J_2(x_i, y_j)$ are replaced by a rank ordering from 1 to m. Here (and throughout this paper) we assume that the J_1 entries in each column are distinct and the J_2 entries in each row are also distinct (i.e. the game is strictly ordinal). Let the pairs of entries in the MO-game be labeled $\{J_1^{co}(x_i, y_j), J_2^{ro}(x_i, y_j)\}$. Clearly, if a PNE $\{x^N, y^N\}$ in the original matrix game exists, it would appear where the pair $(1, 1)$ is located in the MO-game. The maximum number of PNEs that the game can have is therefore bounded by $\mathrm{Min}(n, m)$. Now, since MO-games of a given size are defined in terms of a finite number of preference values, only a finite number of possible MO-games of a given size exist. For example, to determine the total number of different 2×2 MO-games, consider the general 2×2 MO-game shown in Fig. 1.

Here the values a, b, c, d and p, q, w, z represent the possible rank ordering of the preferences of P1 and P2, respectively. If the game is represented in its MO form, then these preferences are restricted to take values from the set $\{1, 2\}$. Consequently, there are a total of four possible column-rank ordered matrices J_1^{co} and four possible row rank-ordered matrices J_2^{ro} as follows:

$$J_{1A}^{co} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad J_{1B}^{co} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad J_{1C}^{co} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{or} \quad J_{1D}^{co} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{3}$$

$$J_{2A}^{ro} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad J_{2B}^{ro} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad J_{2C}^{ro} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \quad \text{or} \quad J_{2D}^{ro} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{4}$$

yielding a total of $4^2 = 16$ possible 2×2 MO-games. These 16 MO-games are listed in the Appendix and labeled according to their relationship to the column and row rank-ordered matrices J_1^{co} and J_2^{ro} in (3) and (4). For example, the MO-game labeled 1_B,2_A corresponds to the game where the players use strategies that yield J_{1B}^{co} and J_{2A}^{ro}. Note that the PNEs for these 16 MO-games have been highlighted in bold type.

Fig. 1 A general 2×2 MO-game

		P2	
		y_1	y_2
P1	x_1	a, p	c, w
	x_2	b, q	d, z

By inspection, we can determine that twelve of these games have a unique PNE, two have 2 PNEs, and two have no PNE. This information is very useful. If the payoff entries of both players in 2×2 game are selected randomly, then it follows that the probability that the game will have a unique PNE is 0.750, that it will have (nonunique) 2 PNEs is 0.125 and no PNE is also 0.125. Now, what if the payoffs of one player (say P1) are known but the payoffs of the other player (P2) are selected randomly? Several possibilities will have to be examined. If the payoffs of P1 correspond to J_{1A}^{co} or J_{1C}^{co}, then clearly the games will have only unique PNEs. That is, the probability that the game has a unique PNE is 1.0, and the probabilities of a nonunique PNE or no PNE are 0 and 0 respectively. If, on the other hand, the payoffs of P1 correspond to J_{1B}^{co} or J_{1D}^{co} then the probability that the game will have a unique PNE is 0.50, that it will have 2 PNEs is 0.25 and no PNE is also 0.25. Clearly, the final probabilities of existence of PNEs depend on the distribution of the payoff entries of the player whose payoffs are known. This distribution therefore affects the probabilities of existence or nonexistence of PNEs in the game.

Let us now examine how this argument can be generalized to an $n \times m$ RE-game. For any given strategy choice by P2, there are $n!$ possible arrangements of the preferences for P1. Since P1 can choose each strategy independently, there are a total of $(n!)^m$ possible arrangements of preferences for P1 corresponding to all possible strategy choices by P2. Similarly, there are a total of $(m!)^n$ possible arrangements of preferences for P2 corresponding to all possible strategy choices by P1. Thus, the total number of MO-games of size $n \times m$ is $(n!)^m(m!)^n$. Classifying the PNEs, whether one or both payoffs are random, by examining all MO-games as was done above for the 2×2 case can quickly become unfeasible for games of higher dimensions. For example, in the case of 3×3 RE-games one has to search a total of 46,656 MO-games. In the case when one player's payoffs are known, there are 216 possible MO-games that need to be considered by that player in determining the probabilities. Clearly a more efficient search approach is needed. In the next section, we will introduce the concept of top-rated preferences MO-games, which will considerably reduce the search space.

3 Top-Rated Preferences Minimal Ordinal Games

One of the main disadvantages in considering all possible MO-games in the search for PNEs is the high degree of redundancy that exists in classifying these games. As an illustration, consider the two 3×3 MO-games shown in Fig. 2. Note that while these two MO-games are different, the top-rated preferences (or TRPs) for both players (i.e. the 1's in J_1^{co} and J_2^{ro}), are located identically in both games and the strategy pair $\{x_1, y_1\}$ is a unique PNE in both games. Since the TRPs for each player in a given situation are the only pieces of information necessary to determine the PNEs, the other preferences can therefore be ignored. For example, in a 3×3 game, each of the nine column-ordered matrices for P1 can be duplicated for the three possible locations of the top preference in the first column, yielding a total of 27 possible alternatives for arranging the TRPs of P1. Likewise, there are 27 possible alternatives for arranging the TRPs of P2. So in order to find the number of games that have no

		P2		
		y_1	y_2	y_3
P1	x_1	1, 1	3, 3	2, 2
	x_2	3, 2	2, 3	3, 1
	x_3	2, 1	1, 3	1, 2

		P2		
		y_1	y_2	y_3
P1	x_1	1, 1	3, 2	3, 3
	x_2	2, 3	2, 2	2, 1
	x_3	3, 1	1, 2	1, 3

Fig. 2 Two 3×3 Mo-games with the same top-rated preferences

PNE or r PNEs we need to consider only $27^2 = 729$ games. We will refer to these games as Top-Rated Preferences MO-games, or TRP-MO-games. Note that for 3×3 games the number of TPR-MO games is much smaller that the number of MO-games (729 vs. 46,656), and in the case where one player's payoffs are known there are only 27 possible games by that player that need to be considered in determining the probabilities. This corresponds to a 64-fold reduction in the search when both payoffs are random and 8-fold reduction when only one player's payoffs are random. As in the 2×2 case, all 729 different 3×3 TRP-MO-games can be listed (see [13] for a complete listing). By inspection of these games, we can determine that 156 have no PNE, 423 have a unique PNE, 144 have 2 PNEs, and 6 have 3 PNEs. Thus, for a 3×3 RE-game the probabilities of existence of PNEs are as follows: The probability for no PNE is 0.21399, for a unique PNE is 0.58025, for 2 PNEs is 0.19753, and for 3 PNEs is 0.00823. In the case where only the payoffs of P2 are randomly selected, a similar approach can be used to determine these probabilities for each of the 27 possible arrangements of TRPs for P1.

In order to generalize the above, we first need to determine the number of TRP-MO-games in an $n \times m$ RE-game. Note that for P1, there is a TRP (i.e. a 1) in each column of his matrix J_1^{co} in any of the n rows. Since J_1^{co} has m columns, P1 can have his TRPs positioned in n^m ways. Likewise P2 can have his TRPs positioned in m^n ways. Thus, the total number of TRP-MO games of size $n \times m$ is $N_{n\times m}^{R\times R} = (n^m)(m^n)$. The $R \times R$ superscript in $N_{n\times m}^{R\times R}$ denotes that the payoffs of both players are randomly selected. If one player's payoffs are known (say P1), then for each of the n^m positions of his TRPs, a search has to be done for the distribution of PNEs within the $N_{n\times m}^{F\times R} = m^n$ possible TRP-MO games that exist. The superscript $F \times R$ in $N_{n\times m}^{F\times R}$ indicates that the payoffs of P1 are known (or fixed) while the payoffs of P2 are randomly selected. In the next section, we will exploit this concept of TRP-MO games to determine the probability distribution of the number of PNEs in OPRE-games.

4 Characterization of Pure Nash Equilibria in $n \times m$ OPRE-Games

Without loss of generality let us assume that in the $n \times m$ OPRE-game the payoffs of P1 (i.e. $J_1(x_i, y_j)$) are known and the payoffs of P2 (i.e. $J_2(x_i, y_j)$) are randomly selected. In the previous section, we showed that for each of the n^m TRPs for P1, a search for the distribution of the PNEs within the m^n resulting TRP-MO games must be carried out. Let $N_{n\times m}^{F\times R}(0)$ and $N_{n\times m}^{F\times R}(r)$ denote the number of TRP-MO games with no PNE and r PNEs respectively. Then:

$$P_{n\times m}^{F\times R}(0)=\frac{1}{m^n}N_{n\times m}^{F\times R}(0), \tag{5a}$$

$$P_{n\times m}^{F\times R}(r)=\frac{1}{m^n}N_{n\times m}^{F\times R}(r), \quad r=1,\dots,\mathrm{Min}(n,m). \tag{5b}$$

The following theorem provides a mechanism for computing these probabilities by first computing $N_{n\times m}^{F\times R}(0)$ and $N_{n\times m}^{F\times R}(r)$.

Theorem 4.1 *Consider an $n\times m$ two player nonzero sum matrix game with no repeated entries. If the payoffs of $P1$ are known and the payoffs of $P2$ are randomly selected, then the probability that the game has no PNE is*

$$P_{n\times m}^{F\times R}(0)=\frac{1}{m^n}\prod_{i=1}^{n}(m-k_i), \tag{6}$$

and the probability that the game has exactly r PNEs is

$$P_{n\times m}^{F\times R}(r)=\frac{1}{m^n}\sum_{j=1}^{m}\left\{\sum_{u\in\Omega}\left[\prod_{\substack{i=1\\ i\neq w_j\\ i\notin S_j^u}}^{n}(m-k_i)\right]\right\}, \tag{7}$$

where k_i is the number of TRPs in row i, w_j is the row number of the jth TRP, v_j is the number of rows greater than w_j in J_1^{co} containing TRPs such that $v_j\neq 0$, $U_j=\binom{v_j}{r-1}$ for $j=1,\dots,m$, $\Omega=\{U_j, j=1,\dots,m$ such that U_j is defined$\}$, and S_j^u for $u\in\Omega$ is a set containing $r-1$ elements $\{s_j^u(1),\dots,s_j^u(t),\dots,s_j^u(r-1)\}$ whose t^{th} element (i.e. $s_j^u(t)$) is the row number of each TRP that satisfies $s_j^u(t)>w_j$ for $t=1,\dots,r-1$ and $u\in\Omega$.

Proof First, apply the TRP-MO simplification to J_1 to get J_1^{co}. Note that J_1^{co} will have one TRP per column, for a total of m TRP entries. For each row $i=1,\dots,n$ of J_1^{co} the number of P2 TRP-MO games that will result in games with no PNE corresponds to the number of columns where J_1^{co} does not have a TRP, i.e. $(m-k_i)$. Since the TRP in row i of J_2^{ro} is independent of the TRP in any other row, the number of TRP-MO games with no PNE is equal to

$$N_{n\times m}^{F\times R}(0)=\prod_{i=1}^{n}(m-k_i)$$

and (6) follows directly from (5a). Now for an arbitrary r such that $1\le r\le \mathrm{Min}(n,m)$ and for the j^{th} TRP, the set S_j^u for $u\in\Omega$ represents one combination of the $r-1$ other rows of J_1^{co} containing TRPs. So for the jth TRP we sum over each set S_j^u for each of the U_j combinations. For each iteration in the sum, we add the number of P2 TRP-MO games that do not create another PNE. That is $N_{n\times m}^{F\times R}(0)$ with $i\neq w_j$ and $i\notin S_j^u$.

This yields

$$N_{n\times m}^{F\times R}(r)=\sum_{j=1}^{m}\left\{\sum_{u\in\Omega}\left[\prod_{\substack{i=1\\ i\neq w_j\\ i\notin S_j^u}}^{n}(m-k_i)\right]\right\}$$

and (7) follows directly from (5b). □

Remark 4.1 In many situations, especially when n and m are large, one might be interested more in knowing the probabilities of existence of unique and nonunique PNEs rather than r PNEs. If we denote the probability of existence of nonunique PNEs by $P_{n\times m}^{F\times R}(r>1)$, then clearly

$$P_{n\times m}^{F\times R}(r>1)=\frac{1}{m^n}\sum_{r=2}^{\mathrm{Min}(n,m)}N_{n\times m}^{F\times R}(r)=\sum_{r=2}^{\mathrm{Min}(n,m)}P_{n\times m}^{F\times R}(r). \tag{8}$$

We note that once $P_{n\times m}^{F\times R}(0)$ and $P_{n\times m}^{F\times R}(1)$ have been determined using (6), and (7) with $r=1$, the probability of nonunique PNEs, can be easily computed as $P_{n\times m}^{F\times R}(r>1)=1-(P_{n\times m}^{F\times R}(1)+P_{n\times m}^{F\times R}(0))$.

We now present an example to illustrate the above results.

Example 4.1 Consider a 4×4 matrix game in which the payoffs of P1 are known and the payoffs of P2 are randomly selected. Assume that the TRPs of P1 are located as illustrated in the following J_1^{co} matrix:

$$J_1^{co}=\begin{bmatrix}1 & 1 & \times & \times\\ \times & \times & \times & 1\\ \times & \times & 1 & \times\\ \times & \times & \times & \times\end{bmatrix}, \tag{9}$$

where the 1 entries represent the TRPs and the $\times$ entries represent lower rated preferences. The row numbers of the TRPs are $w_1=1$, $w_2=1$, $w_3=3$, and $w_4=2$, and $k_1=2$, $k_2=1$, $k_3=1$, and $k_4=0$. The probability $P_{4\times 4}^{F\times R}(0)$ can be easily computed from (6),

$$P_{4\times 4}^{F\times R}(0)=\frac{1}{4^4}\prod_{i=1}^{4}(4-k_i)=0.28125.$$

That is, the probability that this game will have no PNE is 0.28125.

For $r=1$, it follows that $U_j=1$ for $j=1,\dots,4$. Therefore, (7) reduces to

$$P_{4\times 4}^{F\times R}(1)=\frac{1}{4^4}\sum_{j=1}^{4}\left\{\prod_{\substack{i=1\\ i\neq w_j}}^{4}(m-k_i)\right\}=0.46875.$$

Table 1 Probabilities of PNEs in a 4×4 game

Number of PNE	Probabilities when both players have randomly selected payoffs	Probabilities when P1 has TRPs as in (9) and P2 has randomly selected payoffs
No PNE	0.25816	0.28125
1 PNE	0.50635	0.46875
2 PNEs	0.21314	0.21875
3 PNEs	0.02198	0.03125
4 PNEs	0.00037	0.00000

For $r = 2$, we compute $v_1 = 2$, $v_2 = 2$, $v_3 = 0$, and $v_4 = 1$. This yields $U_1 = 2$, $U_2 = 2$, and $U_4 = 1$ (U_3 is undefined since $v_3 = 0$) and $\Omega = \{2, 2, 1\}$. The sets S_j^u for $u \in \Omega$ are $S_1^1 = \{2\}$, $S_1^2 = \{3\}$, $S_2^1 = \{2\}$, $S_2^2 = \{3\}$, and $S_4^1 = \{3\}$. Now, the probability $P_{4\times 4}^{F\times R}(2)$ can be easily computed from (7),

$$P_{4\times 4}^{F\times R}(2) = \frac{1}{4^4} \sum_{j=1}^{4} \left\{ \sum_{u\in\Omega} \left[\prod_{\substack{i=1 \\ i\neq w_j \\ i\notin S_j^u}}^{4} (4 - k_i) \right] \right\} = 0.21875.$$

For $r = 3$, we again use $v_1 = 2$, $v_2 = 2$, $v_3 = 0$, and $v_4 = 1$. This yields $U_1 = 1$ and $U_2 = 1$ (U_3 and U_4 are undefined since $v_3 = 0$ and $v_4 = 1$) and $\Omega = \{1, 1\}$. The sets S_j^u for $u \in \Omega$ are $S_1^1 = \{2, 3\}$ and $S_2^1 = \{2, 3\}$, and the probability $P_{4\times 4}^{F\times R}(3)$ can again be computed from (7),

$$P_{4\times 4}^{F\times R}(3) = \frac{1}{4^4} \sum_{\substack{j=1 \\ w \text{ for each TRP}}}^{4} \left\{ \sum_{u\in\Omega} \left[\prod_{\substack{1 \\ i\neq w \\ i\notin S_j^u}}^{4} (4 - k_i) \right] \right\} = 0.03125.$$

Finally for $r = 4$, we have $v_1 = 2$ which yields $U_1 = \binom{2}{3}$ which is undefined. Thus, the probability that the game will have 4 PNEs will be $P_{4\times 4}^{F\times R}(4) = 0$ (this is also obvious from the fact that only three rows of J_1^{co} contain TRPs).

Note that if the payoffs of both players in a 4×4 game are randomly selected, then the probabilities of the game having PNEs of different multiplicities can also be computed [8, 13]. These, as well as the probabilities when P1 has known payoffs with TRPs as in (9) and P2 has randomly selected payoffs are listed in Table 1.

5 Conclusions

In this paper, we considered nonzero-sum $n \times m$ matrix games where the payoff entries of one player (P1) are known and the payoff entries of the other player (P2) are randomly selected. We determined an expression for the probabilities of a game of this type having no pure Nash equilibria and r pure Nash equilibria. We determined these expressions by partitioning the space of top-rated preferences minimal ordinal games into mutually exclusive spaces of each type. We also expressed the results in terms of the probabilities of existence of a unique pure Nash equilibrium, nonunique pure Nash equilibria, and no pure Nash equilibria. An example was also presented to illustrate the results.

Appendix

All 16 possible different 2×2 MO-games:

$1_A, 2_A$		P2	
		y_1	y_2
P1	x_1	**1, 1**	1, 2
	x_2	2, 1	2, 2

$1_A, 2_B$		P2	
		y_1	y_2
P1	x_1	**1, 1**	1, 2
	x_2	2, 2	2, 1

$1_A, 2_C$		P2	
		y_1	y_2
P1	x_1	1, 2	**1, 1**
	x_2	2, 2	2, 1

$1_A, 2_D$		P2	
		y_1	y_2
P1	x_1	1, 2	**1, 1**
	x_2	2, 1	2, 2

$1_B, 2_A$		P2	
		y_1	y_2
P1	x_1	**1, 1**	2, 2
	x_2	2, 1	1, 2

$1_B, 2_B$		P2	
		y_1	y_2
P1	x_1	**1, 1**	2, 2
	x_2	2, 2	**1, 1**

$1_B, 2_C$		P2	
		y_1	y_2
P1	x_1	1, 2	2, 1
	x_2	2, 2	**1, 1**

$1_B, 2_D$		P2	
		y_1	y_2
P1	x_1	1, 2	2, 1
	x_2	2, 1	1, 2

$1_C, 2_A$		P2	
		y_1	y_2
P1	x_1	2, 1	2, 2
	x_2	**1, 1**	1, 2

$1_C, 2_B$		P2	
		y_1	y_2
P1	x_1	2, 1	2, 2
	x_2	1, 2	**1, 1**

$1_C, 2_C$		P2	
		y_1	y_2
P1	x_1	2, 2	2, 1
	x_2	1, 2	**1, 1**

$1_C, 2_D$		P2	
		y_1	y_2
P1	x_1	2, 2	2, 1
	x_2	**1, 1**	1, 2

$1_D, 2_A$		P2	
		y_1	y_2
P1	x_1	2, 1	1, 2
	x_2	**1, 1**	2, 2

$1_D, 2_B$		P2	
		y_1	y_2
P1	x_1	2, 1	1, 2
	x_2	1, 2	2, 1

$1_D, 2_C$		P2	
		y_1	y_2
P1	x_1	2, 2	**1, 1**
	x_2	1, 2	2, 1

$1_D, 2_D$		P2	
		y_1	y_2
P1	x_1	2, 2	**1, 1**
	x_2	**1, 1**	2, 2

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