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An Inequality for the Volume of a Tetrahedron

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Abstract. In this note we prove a curious inequality involving the sides and volume of a tetrahedron.

There are many inequalities involving the area and sides of a triangle, but analogous inequalities for tetrahedra are less common. The goal of this short note is to prove the following inequality involving the volume and sides of a tetrahedron.

Theorem 1. *Let V be the volume of a tetrahedron $ABCD$ and let $a = AB \cdot CD$, $b = AC \cdot BD$, $c = AD \cdot BC$. Then*

$$(a+b-c)(a+c-b)(b+c-a) \geq 72V^2 \tag{1}$$

and equality holds if and only if the tetrahedron is equifacial.

Recall that a tetrahedron $ABCD$ is **equifacial** (or **isosceles**) if its faces are all congruent triangles. Equivalently, $ABCD$ is equifacial if and only if $AB = CD$, $AC = BD$, and $AD = BC$. There are many equivalent characterizations of equifacial tetrahedra (see [**1**, IV.6.b], [**3**], [**4**, Chapter 9], [**5**], [**6**]) and Theorem 1 can be considered as yet another such characterization.

We will derive Theorem 1 from two classical results in solid geometry. The first of them is the following theorem due to August Leopold Crelle [**2**]. (A. L. Crelle (1780–1855) was a German mathematician and the founder of the *Journal für Die Reine and Angewandte Mathematik*, commonly known as *Crelle's Journal*, which has been one of the leading mathematical journals ever since its establishment in 1826.)

Crelle's Theorem. *Let V and R be the volume and the circumradius of a tetrehedron $ABCD$, respectively. Then the quantities $a = AB \cdot CD$, $b = AC \cdot BD$, $c = AD \cdot BC$ are side-lengths of a triangle whose area S is given by the formula $S = 6RV$.*

The second result needed to prove Theorem 1 is the following proposition.

Proposition 2. *Let G be the centroid of a tetrahedron $ABCD$. Then, for any point P, we have*

$$4(PA^2 + PB^2 + PC^2 + PD^2) = AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 + 16PG^2.$$

Both Crelle's theorem and Proposition 2, as well as numerous other properties of equifacial tetrahedra and many other interesting results from solid geometry, can be found in the wonderful book [**6**] (see Problems 302, 297, 21). Unfortunately, this book

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is very hard to find. We will outline proofs (different from those in [6]) of both results at the end of this note.

We are ready now to prove Theorem 1. Applying Heron's formula to the area S in Crelle's theorem, we get $(a+b+c)(a+b-c)(a+c-b)(b+c-a) = 16S^2 = 16(6RV)^2$. It follows easily from this equality that Theorem 1 is equivalent to the following result, which is of interest in its own right.

Theorem 3. *Let R be the circumradius of a tetrahedron $ABCD$ and let $a = AB \cdot CD$, $b = AC \cdot BD$, $c = AD \cdot BC$. Then*

$$8R^2 \geq a + b + c \tag{2}$$

and equality holds if and only if the tetrahedron is equifacial.

It remains to prove Theorem 3. Taking for P in Proposition 2 the circumcenter O of $ABCD$, we get the equality $16R^2 = AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 + 16OG^2$. Since $(AB - CD)^2 \geq 0$, we have $AB^2 + CD^2 \geq 2a$. Similarly, $AC^2 + BD^2 \geq 2b$ and $AD^2 + BC^2 \geq 2c$. Adding these inequalities, we get $16R^2 \geq 2(a+b+c) + 16OG^2$, which proves inequality (2). Furthermore, it is clear that equality in (2) holds if and only if $O = G$, $AB = CD$, $AC = BD$, and $AD = BC$. To complete the proof of Theorem 3 it suffices to show that $O = G$ for any equifacial tetrahedron. Indeed, Proposition 2 applied to any vertex P of an equifacial tetrahedron $ABCD$ yields $4(a+b+c) = 2(a+b+c) + 16PG^2$. It follows that G is equidistant from all vertices of $ABCD$, i.e., $G = O$. This completes our proofs of Theorems 3 and 1. Perhaps we should mention that a tetrahedron is equifacial if and only if its centroid coincides with its circumcenter (see [1, sec. 298], [3], [6, Problem 304]).

We end this note with an outline of proofs of Crelle's theorem and of Proposition 2.

To outline a proof of Crelle's theorem, we need to recall the notion of an inversion. An **inversion** I with center O and radius r assigns to any point $P \neq O$ a point $Q = I(P)$ on the ray OP such that $OP \cdot OQ = r^2$. Inversion preserves angles (is conformal), maps circles (spheres) not passing through O to circles (spheres) not passing through O and circles (spheres) passing through O to lines (planes) not passing through O. Furthermore, for any points A, B, we have

$$I(A)I(B) = \frac{AB \cdot r^2}{OA \cdot OB}. \tag{3}$$

For more details about inversions, we refer to [1, Chapter VIII] and [7, Section 8.5].

To prove Crelle's theorem, consider the inversion I with center at the vertex D and radius $r = \sqrt{DA \cdot DB \cdot DC}$. By (3), the triangle $I(A)I(B)I(C)$ has sides a, b, c. In order to compute the area S of this triangle, consider the volume V^* of the tetrahedron $DI(A)I(B)I(C)$. The image under I of the circumsphere of $ABCD$ is the plane $I(A)I(B)I(C)$. Let X be the point on the circumsphere of $ABCD$ diametrically opposite point D. Then DX is perpendicular to the circumsphere and, since I is conformal, $DI(X)$ is perpendicular to the plane $I(A)I(B)I(C)$; hence the distance from D to the plane $I(A)I(B)I(C)$, which is $DI(X)$, is equal to $r^2/2R$. Thus $V^* = Sr^2/6R$. On the other hand, since $DI(A)I(B)I(C)$ is obtained from $DABC$ by rescaling the edges DA, DB, DC, we have

$$\frac{V^*}{V} = \frac{DI(A) \cdot DI(B) \cdot DI(C)}{DA \cdot DB \cdot DC} = \frac{r^6}{DA^2 \cdot DB^2 \cdot DC^2} = r^2.$$

It follows that $Sr^2/6RV = r^2$, i.e., $S = 6RV$.

Proposition 2 is a special case of a classical result in mass point geometry due to Lagrange (and many others, who discovered it independently). Consider points $A_1, \dots, A_n$ and let G be their center of mass, i.e., $\sum_{i=1}^n \overrightarrow{GA_i} = 0$. For any point P define the **moment of inertia** I_P of the points $A_1, \dots, A_n$ with respect to P as $I_P = \sum_{i=1}^n PA_i^2$. Then

$$I_P = \sum_{i=1}^n (\overrightarrow{PG} + \overrightarrow{GA_i})^2 = \sum_{i=1}^n GA_i^2 + nPG^2 + 2\overrightarrow{PG} \cdot \sum_{i=1}^n \overrightarrow{GA_i} = I_G + nPG^2. \tag{4}$$

Adding the equalities (4) for $P = A_1, \dots, A_n$, we get

$$2\sum_{i<j} A_iA_j^2 = \sum_{i=1}^n I_{A_i} = nI_G + n\sum_{i=1}^n A_iG^2 = 2nI_G. \tag{5}$$

Multiplying (4) by n and using (5), we obtain the following equality:

$$nI_P = \sum_{i<j} A_iA_j^2 + n^2PG^2. \tag{6}$$

Proposition 2 is a special case of (6), when $n = 4$, $A_1 = A$, $A_2 = B$, $A_3 = C$, and $A_4 = D$.

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