THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XI. DECEMBER, 1904. No. 12.

THE TRANSCENDENCE OF π AND e.

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§1. The proof that π is a transcendental number is ordinarily arranged as follows. If π should satisfy any algebraic equation, so would $\pi.\sqrt{-1}$. But it is well known that

$$e^{\pi.\sqrt{-1}}=-1 \qquad (A).$$

Hence if $\pi.\sqrt{-1}$ is one of the m roots $z_1, z_2, \ldots\ldots, z_m$, of an algebraic equation, we must have

$$(e^{z_1}+1)(e^{z_2}+1)\ldots\ldots(e^{z_m}+1)=0 \qquad (B)$$

since one of its factors is zero. On expanding (B) we obtain

$$c+e^{x_1}+e^{x_2}+\ldots\ldots+e^{x_n}=0 \qquad (C)$$

where $x_1, x_2, \ldots\ldots, x_n$ are the n roots of an algebraic equation and where c is a whole number not zero. The rest of the argument consists in showing that equation (C) is impossible.

The proof* that (C) is impossible is so difficult for most students that it

*The principal references in English on the subject of the transcendence of π and e seem to be the translation by W. W. Beman of the chapter on Transcendental Numbers in Weber's *Algebra* published in the *Bulletin of the American Mathematical Society*, Vol. 3 (1897), p. 174, and the translation by Beman and Smith of Klein's *Famous Problems of Elementary Geometry* (Ginn & Co., Boston). A good elementary treatment in the German language is that by Weber and Wellstein, *Encyclopadie der Elementarmathematik*, Vol. I, pp. 418-432. (B. G. Teubner, Leipzig).

seems worth while to publish the simplified arrangement of the argument that is given below. The simplification consists in leaving out one factor ordinarily multiplied into the function $\phi(x)$ and in the device of adding together the terms of equation (3) first by diagonals and then by columns.

§2. Our task is to show that

$$c+e^{x_1}+e^{x_2}+\dots\dots+e^{x_n} \tag{1}$$

cannot be zero if c *is an integer not zero and* $x_1, x_2, \dots\dots, x_n$ *are the roots of an equation*

$$f(x)=a_0+a_1x+a_2x^2+\dots\dots+a_nx^n=0 \tag{2}$$

with integral coefficients, $a_0\neq 0$, $a_n\neq 0$.

The scheme of proof is to find a number N such that when we multiply it into (1) the resulting expression becomes equal to a whole number plus a quantity numerically less than unity, a sum which surely cannot be zero. To find this multiplier N, we study the series for e^{x_k} where x_k is any one of the roots of $f(x)=0$.

$$e^{x_k}=1+\frac{x_k}{1!}+\frac{x_k^2}{2!}+\frac{x_k^3}{3!}+\dots\dots$$

Multiplying this successively by arbitrary factors, we obtain the equations called (3):

$$e^{x_k}.1!.b_1=b_1.1!+b_1x_k(1+\frac{x_k}{2}+\frac{x_k^2}{2.3}+\dots\dots)$$

$$e^{x_k}.2!.b_2=b_2.2!(1+\frac{x_k}{1!})+b_2x_k^2(1+\frac{x_k}{3}+\frac{x_k^2}{3.4}+\dots\dots)$$

$$e^{x_k}.3!.b_3=b_3.3!(1+\frac{x_k}{1}+\frac{x_k^2}{2!})+b_3x_k^3(1+\frac{x_k}{4}+\frac{x_k^2}{4.5}+\dots\dots)$$

.... — ....

$$e^{x_k}.s!.b_s=b_s.s!(1+\frac{x_k}{1!}+\frac{x_k^2}{2!}+\dots+\frac{x_k^{s-1}}{(s-1)!})+b_sx_k^s(1+\frac{x_k}{s+1}+\frac{x_k}{(s+1)(s+2)}+\dots)$$

Now $b_1, \dots\dots, b_s$ can be regarded as coefficients of an arbitrary polynomial

$$\phi(x)=b_0+b_1x+b_2x^2+\dots\dots+b_sx^s.$$

Differentiating, we have

$$\phi'(x)=b_1+b_2.2.x+\dots\dots+b_s.s.x^{s-1},$$

and in general

$$\phi^{(m)}(x)=b_m.m!+b_{m+1}.\frac{(m+1)!}{1!}x+\ldots\ldots+b_s.\frac{s!}{(s-m)!}x^{s-m}.$$

If we add together the equations (3), we evidently obtain as the sum of the terms in the main diagonal, from $b_1 1!$ to $b_s.s!.\frac{x_k^{s-1}}{(s-1)!}$, the polynomial $\phi'(x_k)$; as the sum of the terms in the next lower diagonal $\phi''(x_k)$, etc. We therefore have

$$e^{x_k}(1!b_1+2!b_2+\ldots+s!b_s)=\phi'(x_k)+\phi''(x_k)+\ldots+\phi^{(s)}(x_k)+\sum_{m=1}^{s}b_m x_k^m R_{km} \quad (4)$$

in which $R_{km}=1+\frac{x_k}{m+1}+\frac{x_k^2}{(m+1)(m+2)}+\ldots\ldots$

Suppose now that $\phi(x)$, which is perfectly arbitrary, be chosen as below so that

$$\phi'(x_k)=0, \quad \phi''(x_k)=0, \ldots\ldots, \phi^{(p-1)}(x_k)=0,$$

for every x_k, $p<s$. By returning to the arrangement of (3) and leaving out the terms due to $\phi'(x_k)$, ..., $\phi^{(p-1)}(x_k)$, we could then rewrite (4) in the form

$$e^{x_k}(1!b_1+2!b_2+\ldots\ldots+s!b_s)=\sum_{m=1}^{s}b_m x_k^m R_{km}$$

$$+b_p.p!$$

$$+b_{p+1}.(p+1)!(1+\frac{x_k}{1!})$$

$$+b_{p-2}.(p+2)!(1+\frac{x_k}{1!}+\frac{x_k^2}{2!})+\ldots\ldots$$

$$+b_s.s!(1+\frac{x_k}{1!}+\frac{x_k}{2!}+\ldots+\frac{x_k^{s-p}}{(s-p)!}). \quad (5).$$

A choice of $\phi(x)$ that satisfies the conditions just required is

$$\phi(x)=\frac{x^{p-1}}{(p-1)!}(a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n)^p\equiv\frac{x^{p-1}(f(x))^p}{(p-1)!}$$

of which every x_k is a p-tuple root, by (2). Here p is still perfectly arbitrary, but $s=np+p-1$, the degree of $\phi(x)$. Expanding $\phi(x)$, we find on account of the factor x^{p-1}

$$b_0=0,\ b_1=0,\ \cdots\cdots,\ b_{p-2}=0,$$

$$b_{p-1}=\frac{a_0{}^p}{(p-1)\,!},\quad b_p=\frac{I_p}{(p-1)\,!},\ \cdots\cdots,\ b_s=\frac{I_s}{(p-1)\,!},$$

where I_p, ..., I_s are all integers.

Now the coefficient of e^{x_k} in (5) evidently becomes

$$N_p=a_0{}^p+\frac{I_p}{(p-1)\,!}.\,p\,!+\frac{I_{p+1}}{(p-1)\,!}.(p+1)\,!+\cdots\cdots+\frac{I_s\,s\,!}{(p-1)\,!}.$$

If the arbitrary p is taken as a prime number greater than a_0, this expression is the sum of $a_0{}^p$, which cannot contain p as a factor, plus a number of other integers each of which does contain the factor p. N_p is therefore *not zero and not divisible by p.*

Further, since $(p+k)\,!\div[(p-1)\,!\ k\,!]$ is an integer divisible by p, it follows that all of the coefficients of the last block of terms in (5) contain p as a factor. On adding the columns of (5) we have:

$$N_pe^{x_k}=p[P_0+P_1x_k+P_2x_k{}^2+\cdots\cdots+P_{s-p}(x_k)^{s-p}]+\sum_{m=1}^{s}b_mx_kR_{km}, \tag{6}$$

where P_0, P_1, ..., P_{s-k} are integers.

Before completing our argument we need only to show that by choosing as p a prime number sufficiently large, the last term of (6) can be made as small as we please. If a is a number greater than unity and greater than any of the n roots x_k of $f(x)$,

$$|\,R_{km}\,|=|\,1+\frac{x_k}{m+1}+\frac{x_k{}^2}{(m+1)(m+2)}+\cdots\cdots\,|<|\,1+\frac{a}{1!}+\frac{a^2}{2!}+\cdots\cdots\,|\,.$$

$$\therefore\ |\,R_{km}\,|<e^a\,.$$

Now since the coefficients b_m in (6) are the coefficients of $\phi(x)$ and since each coefficient of $\phi(x)$ is numerically less than or equal to the corresponding coefficient of

$$\frac{x^{p-1}}{(p-1)\,!}(\,|\,a_0\,|+|\,a_1\,|\,x+|\,a_2\,|\,x^2+\cdots\cdots+|\,a_n\,|\,x^n)^p,$$

we have the inequality, Q denoting a constant,

$$|\sum_{m=1}^{s}b_mx_k{}^mR_{km}\,|<e^a\cdot\frac{a^{p-1}}{(p-1)\,!}(\,|\,a_0\,|+|\,a_1\,|\,a+\cdots\cdots+|\,a_n\,|\,a)^p<\frac{(Q)^p}{(p-1)\,!}.$$

The last expression, designated Σ_p, is the pth term of the series for Qe^Q and therefore approaches zero as p is increased indefinitely.

We now choose the arbitrary prime number $p>1$ so that it shall be larger that a_0, larger than C, and also so that $\Sigma_p<1/n$. *The number N_p is the required multiplier N.*

For if we multiply N_p into (1) in follows directly from equation (6) that

$$N_p(C+e^{x_1}+e^{x_2}+\cdots\cdots+e^{x_n})=N_pC+p(P_0+P_1S_1+P_2S_2+\cdots\cdots+P_{s-p}S_{s-p})+r_1+r_2+\cdots\cdots+r_n \quad (7)$$

where $r_k=\sum_{m=1}^{s} b_m(x_k)^m R_{km}<1/n$, $S_i=x^i{}_1+x^i{}_2+\cdots\cdots+x^i{}_n$.

But from Newton's formulas*

$$S_1+a_1=0,\quad S_2+a_1S_1+2a_2=0,\ \cdots\cdots$$

it follows that $S_1, S_2, \cdots\cdots, S_{s-p}$ are whole numbers. Hence the second term of the right-hand member of (7) is an integer divisible by p. On the contrary, N_p and C are not divisible by p. The sum of these terms therefore is a whole number greater than $+1$ or less than -1; and since the sum $r_1+r_2+\cdots\cdots+r_n$ is less than unity the right-hand member of (7) cannot be zero. Hence the left-hand member of (7) is not zero and hence (1) cannot be zero.

§3. The proof that e is a transcendental number can be effected by almost precisely the same argument as that given above. It is required to show that the algebraic equation with integral coefficients

$$c+c_1e+c_2e^2+\cdots\cdots+c_ne^n=0 \quad (1')$$

is impossible. Evidently no generality is lost by assuming $c\neq 0$ and $c_n\neq 0$. Let

$$f(x)=(x-1)(x-2)\cdots\cdots(x-n)=a_0+a_1x+a_2x^2+\cdots\cdots+a_nx^n. \quad (2')$$

The argument now is exactly like that of §2 from equation (2) to the sentence introducing equation (7). At this point we observe that since all the roots of $f(x)$ are integers, (6) may be written

$$N_pe^{x_k}=pW_k+r_k,$$

where W_k is a whole number and r_k is less than $1/n$. We therefore have

$$N_p(c+c_1e+\cdots+c_ne^n)=c.N_p+p(W_1+W_2+\cdots+W_n)+r_1+r_2+\cdots+r_n. \quad (7')$$

In the right-hand member, the first term is not divisible by p, the second term is divisible by p and the third term is numerically less than unity. From this it follows as before that the left-hand member of (7′) cannot be zero and hence that (1′) is impossible. Therefore e cannot satisfy an algebraic equation.

*Cf. Burnside and Panton, *Theory of Equations*, Chapter VIII, or any book on higher algebra.