Integer Sequences and the Nature of Proof

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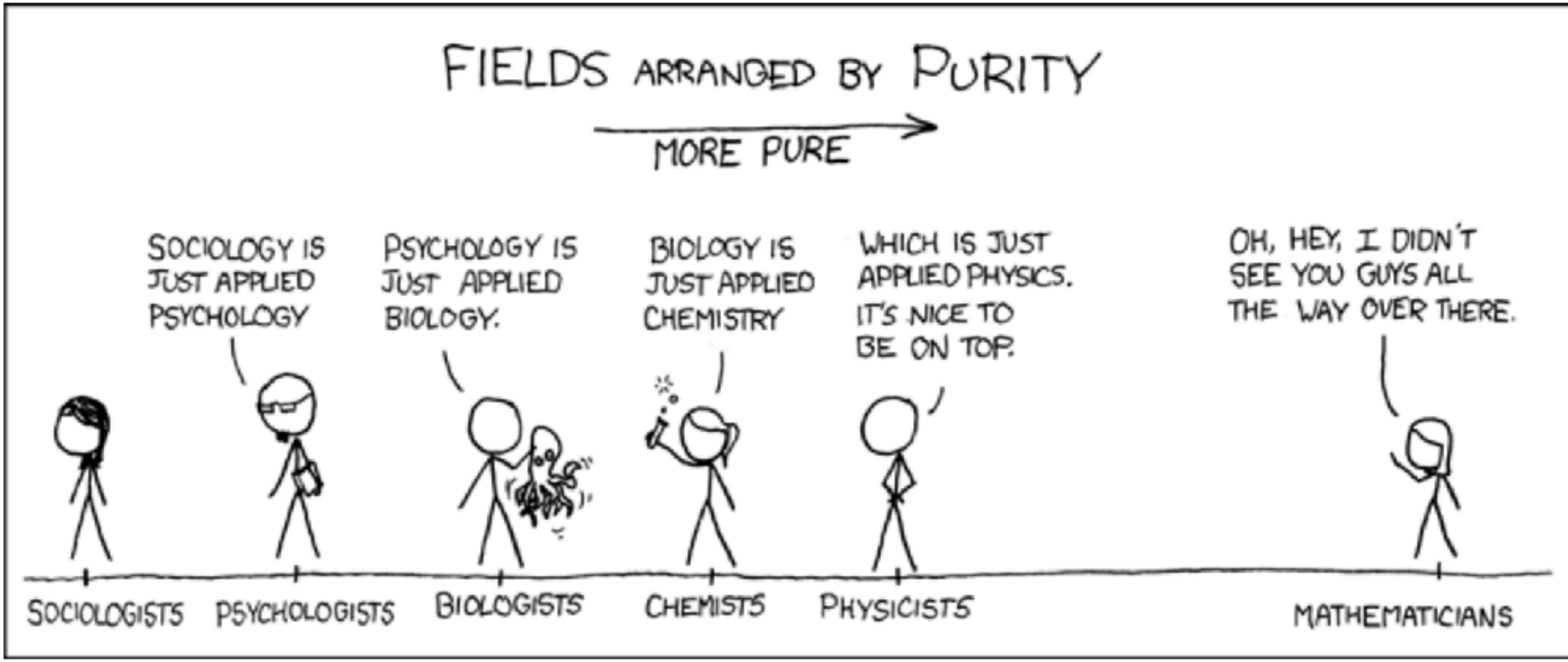
Math 640: Experimental Math Class
Guest Lecture, Jan 25 2018


FIELDS ARRANGED BY PURITY
MORE PURE
SOCIOLOGY IS JUST APPLIED PSYCHOLOGY
PSYCHOLOGY IS JUST APPLIED BIOLOGY.
BIOLOGY IS JUST APPLIED CHEMISTRY
WHICH IS JUST APPLIED PHYSICS. IT'S NICE TO BE ON TOP.
OH, HEY, I DIDN'T SEE YOU GUYS ALL THE WAY OVER THERE.
SOCIOLOGISTS
PSYCHOLOGISTS
BIOLOGISTS
CHEMISTS
PHYSICISTS
MATHEMATICIANS

xkcd.com/435/

Outline

- Riemann Hypothesis (no proofs!)
- Lexicographically earliest seqs. (some proofs)
- Van Eck's seq. and Mr Robot (1 proof, 1 conj.)
- Coordination sequences (many proofs, need help)

Sources for sequences include:

- Binomial coefficient identities: Sum… = f(n) (lots from A=B)
- Arithmetic inequalities: sigma(n) > n+sqrt(n) is A079528
- Lexicographically earliest sequences (e.g. EKG)
- Coordination sequences (chemistry, graph theory)

Sequences in OEIS Related to Riemann Hypothesis (1)

A79526:

a(n) = $\lfloor e^{H(n)} \log H(n) \rfloor - \sigma(n)$

where $$H(n) := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$\sigma(n)$ **= sum of divisors of n**

-1, -2, -1, -2, 2, -2, 4, 0, 4, 2, 10, -3, 13, 6, 9, 4, 20, 2, 23, 4, …

Theorem (Kaneko, Lagarias, Robin):

a(n) > 0 for all n > 50 iff RH is true

Moral: Any arithmetic inequality can be turned into an integer sequence.

Sequences in OEIS Related to Riemann Hypothesis (2)

A57641: $$a(n) = \lfloor H(n) + e^{H(n)} \log H(n) \rfloor - \sigma(n)$$

where $$H(n) := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$\sigma(n)$ **= sum of divisors of n**

0, 0, 1, 0, 4, 0, 7, 2, 7, 5, 13, 0, 17, 9, 12, 8, 23, 5, 27, 8, 21, …

Theorem (Lagarias 2002, Robin 1984):

a(n) >= 0 for all n iff RH is true

Sequences in OEIS Related to Riemann Hypothesis (3)

A2410:

14, 21, 25, 30, 33, 38, 41, 43, 48, 50, 53, 56, 59, 61, 65, 67,

Nearest integer to imaginary part of n-th zero of Riemann zeta function.

```
The imaginary parts of the first 4 zeros are
14.134725... (A058303),
21.0220396... (A065434),
25.01085758... (A065452),
30.424876... (A065453), …
```

See also:
Index to OEIS: Riemann Hypothesis, sequences related to

The three worst non-proofs

in order of increasing badness

It's obvious
It's true for the first 10000 terms
Here is the proof … [and it's wrong]

When you write your paper proving that the long-standing Gauss PQR conjecture is true, start off by describing the previous attempts at proof, and where they failed and then explain how your proof is better

EKG Sequence (A64413)

1, 2, 4, 6, 3, 9, 12, 8, 10, 5, 15, ...

$a(1)=1, \; a(2)=2,$

$a(n)$ = min k such that

- GCD $\{ a(n-1), k \} > 1$
- k not already in sequence

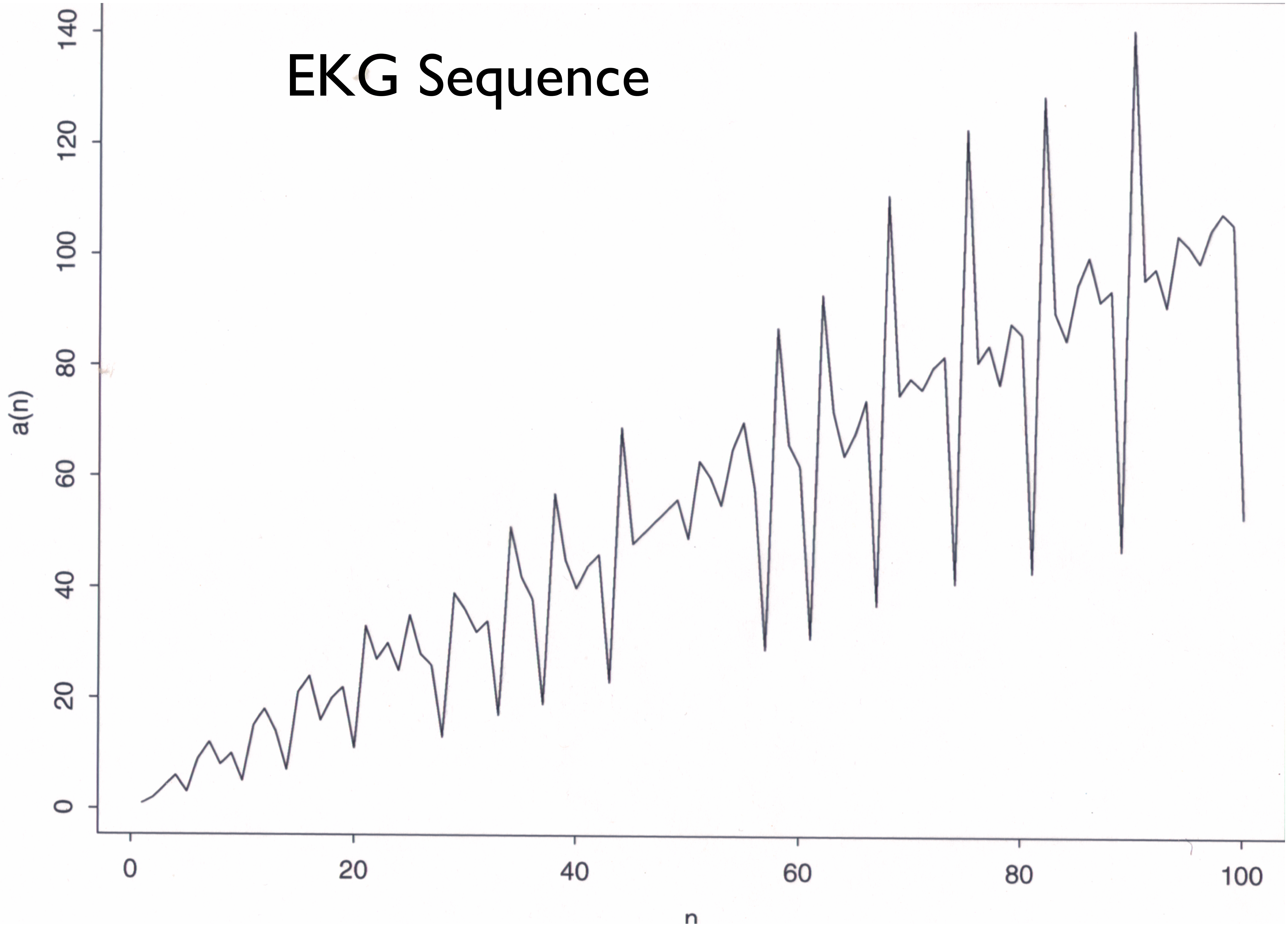
- Jonathan Ayres, 2001

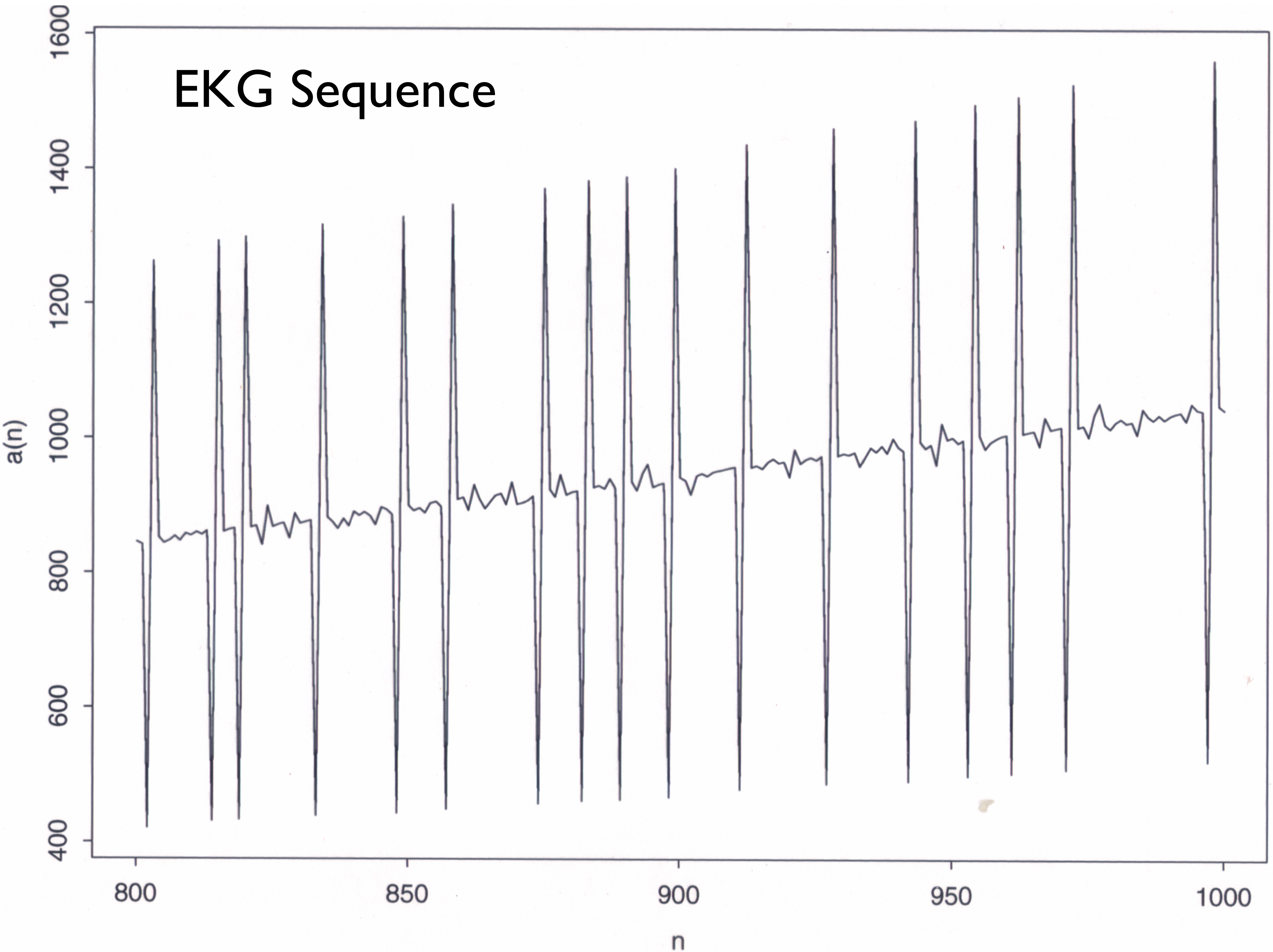
- Analyzed by Lagarias, Rains, NJAS, Exper. Math., 2002

- Gordon Hamilton, Videos related to this sequence:

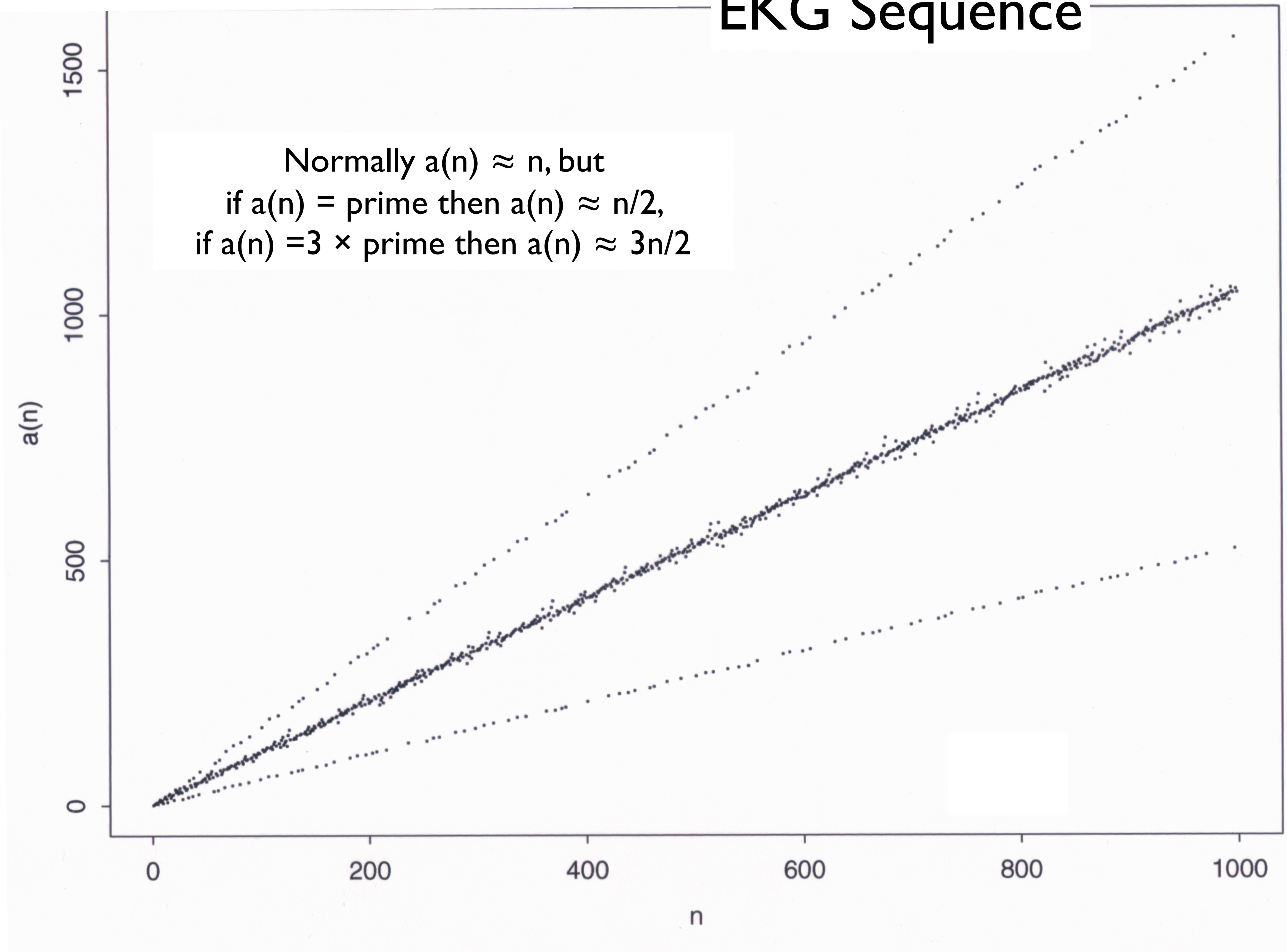
https://www.youtube.com/watch?v=yd2jr30K2R4&feature=youtu.be http://m.youtube.com/watch?v=Y2KhEW9CSOA

EKG Sequence




EKG Sequence
a(n)
n
400
600
800
1000
1200
1400
1600
800
850
900
950
1000

EKG Sequence



Question: Does every number appear?

High school student:
That’s obvious!

Me: I don’t think so!

The EKG sequence (cont)

A64413

Theorem: Every positive number appears

Proof:

There are several steps. (i) Sequence is infinite (easy).
(ii) Let T(m) = n such that a(n)=m, or -1 if m is missing from sequence.
Let W(m) = max T(i), i <= m. Then if n > W(m), a(n) > m.

(iii) Let p = prime. Exists n such that p | a(n). If not, no prime q>p can divide any term either, because if a(n) = qk then pk would be a smaller choice.
So all terms are products just of primes < p.
Choose n>W(p^2), say a(n) = qk, for prime q<p, so qk > p^2.
Then pk < p^2 < qk was a smaller candidate for a(n), contradiction.

(iv) When p first divides a(n), say a(n) = kp, then k is a prime < p.
If k = 2 we have a(n)=2p, a(n+1)=p. Otherwise we have a(n)=kp, a(n)=p, a(n+1)=2p. Either way we see adjacent terms p and 2p.

Proof (continued)

(v) If for some prime p there are infinitely many multiples of p, then all multiples of p are in the sequence. If not, let kp = smallest missing multiple of p. Find n >W(kp) with a(n) = mp. Then kp < mp was a smaller candidate for a(n), a contradiction.

(vi) If for some prime p all multiples of p are in the sequence then all numbers appear. For suppose k is smallest missing number. Find n > W(k) such that a(n) is multiple of kp. Then k was smaller candidate for a(n), contradiction.

(vii) By (iii) and (iv) we see infinitely many multiples of 2, and by (v) and (vi) we see all numbers.

QED

The Yellowstone Permutation

A98550

a(n) = smallest number not yet in seq. such that
gcd(a(n-2), a(n)) >1, gcd(a(n-1),a(n) = 1;
starts 1,2,3

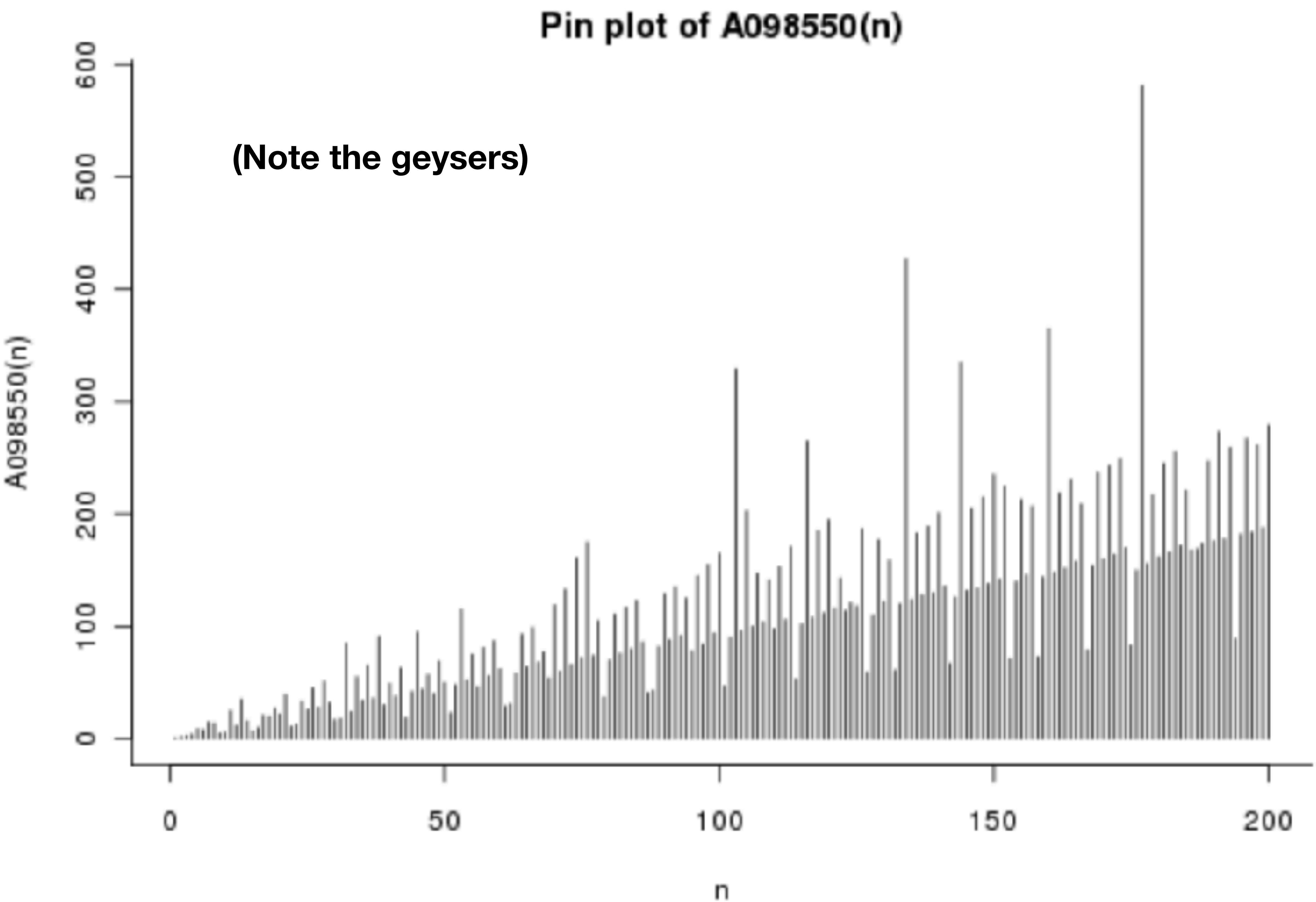
1, 2, 3, 4, 9, 8, 15, 14, 5, 6, 25, 12, 35, 16, 7, 10, 21, 20, 27

Theorem (*): Every positive number appears

(*) Applegate, Havermann, Selcoe, Shevelev, S., Zumkeller, 2015

See A98550 for details

The Yellowstone Permutation



Remy Sigrist's A280864

Lexicographically earliest seq. of distinct numbers such that if p divides a(n) then it divides EITHER a(n-1) or a(n+1) BUT NOT BOTH

1, 2, 4, 3, 6, 8, 5, 10, 12, 9, 7, 14, 16, 11, 22, 18, 15, 20, 24

Conjecture: Every positive number appears

I can prove:
every prime appears;
every prime divides infinitely many odd terms;
every even number appears; etc.

Show every odd number appears!

Help!

Jan Ritsema van Eck's Sequence

```
0, 0, 1, 0, 2, 0, 2, 2, 1, 6, 0, 5,
0, 2, 6, 5, 4, 0, 5, 3, 0, 3, 2, 9,
0, 4, 9, 3, 6, 14, 0, 6, 3, 5, 15, 0,
5, 3, 5, 2, 17, 0, 6, 11, 0, 3, 8, 0, ...
```

a(n): how far back did we last see a(n-1)?
or 0 if a(n-1) never appeared before.

Van Eck: A181391

Mentioned in Guardian interview.

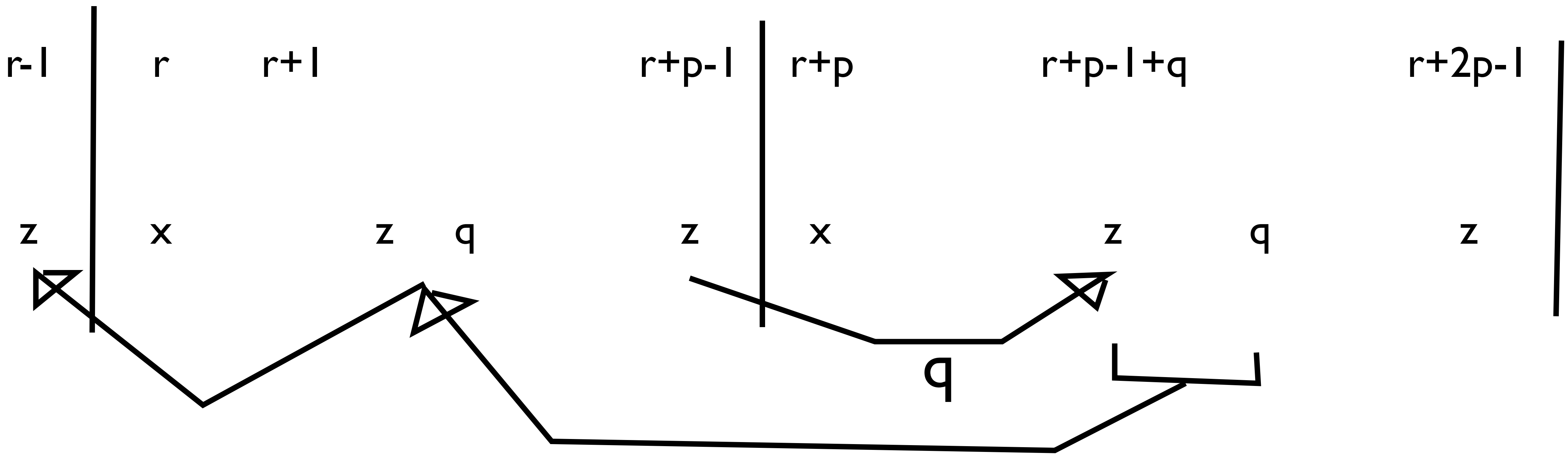
TV series Mr Robot, current 2017/2018 season, has Alternate Reality Game that maybe mentions this sequence?

Thm. (Van Eck) There are infinitely many zeros.

Proof: (i) If not, no new terms, so bounded.
Let M = max term.
Any block of length M determines the sequence.
Only M^M blocks of length M.
So a block repeats.
So sequence becomes periodic.
Period contains no 0's.

Van Eck: A181391

Proof (ii). Suppose period has length p and starts at term r.



Therefore period really began at term r − 1.

....

Therefore period began at start of sequence.

But first term was 0, contradiction.

Van Eck: A181391

It seems that:

lim sup a(n) / n = 1

Gaps between 0's roughly log_10 n

Every number eventually appears

Proofs are lacking!

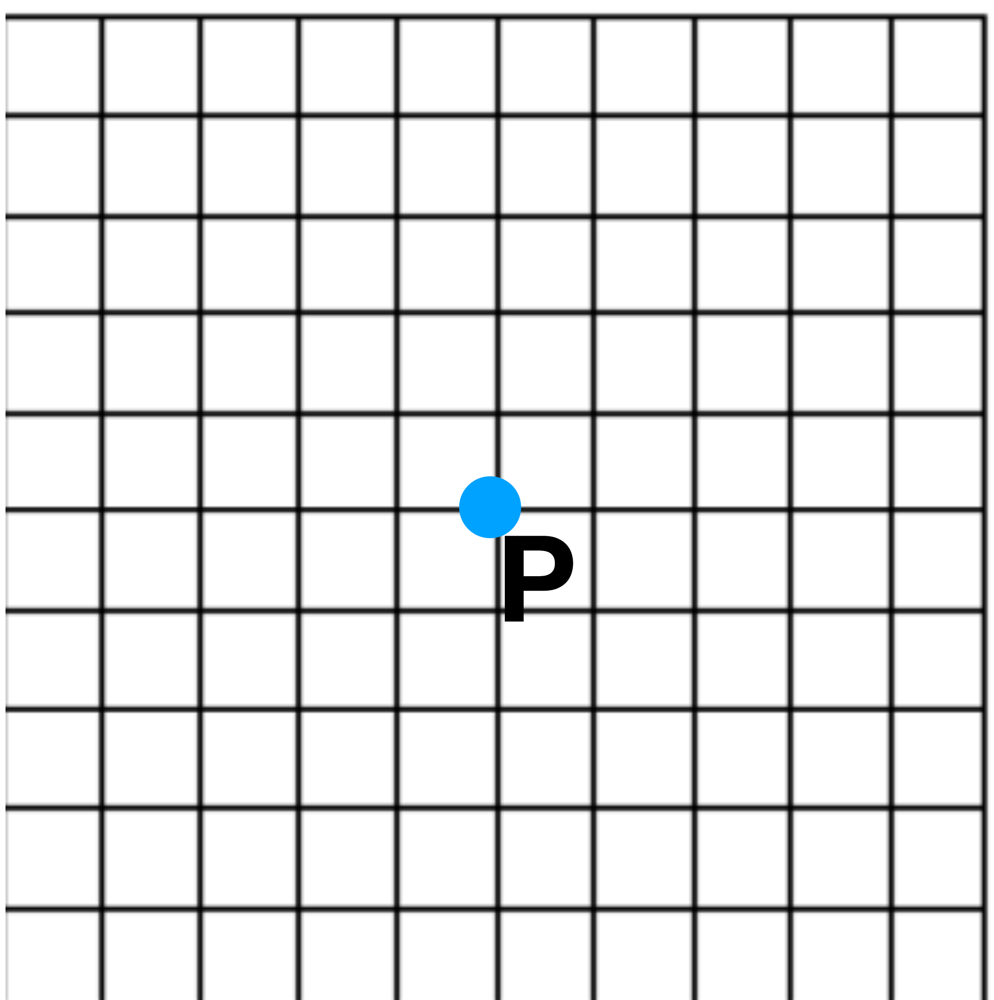
Van Eck: A181391

Coordination Sequences

(Need help with this project - someone interested in sequences, experimenting, guessing, with a Windows machine.
Let me know, njasloane (AT) gmail.com, if interested in helping)

Definition. G = graph, P = node,
the coordination sequence w.r.t P:
a(n) = number of nodes at edge-distance n from P

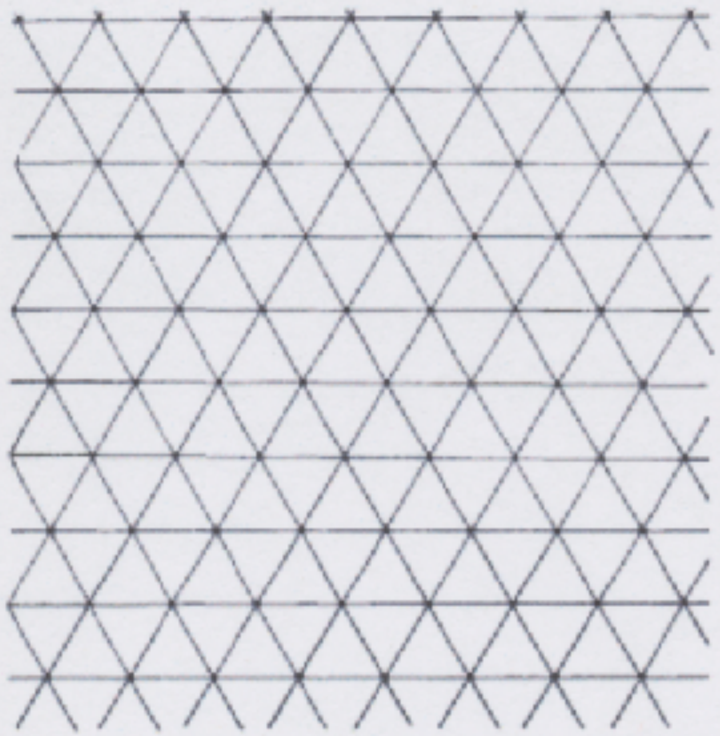
G



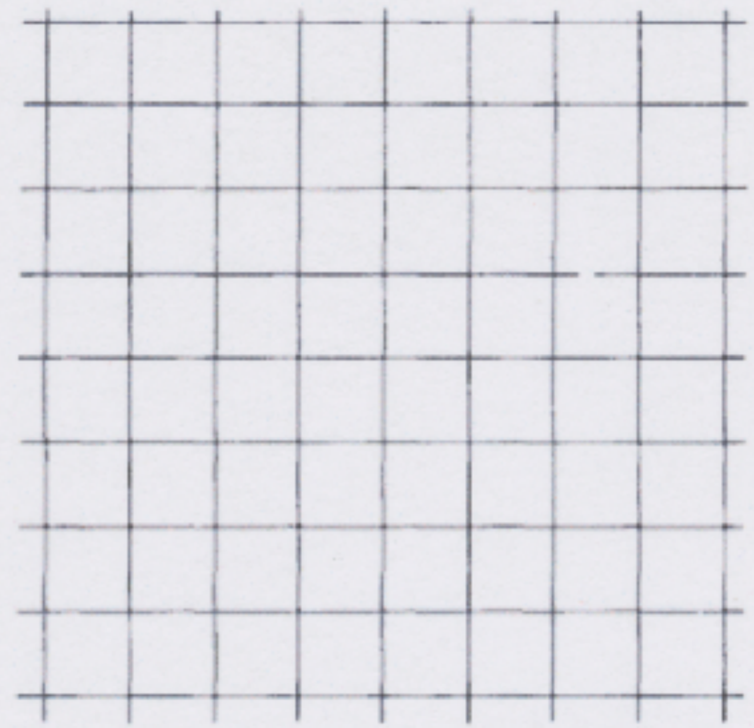
CS is 1, 4, 8, 12, 16, 20, 24, 28, …

A8574

The 11 uniform or Archimedean tilings (part 1)

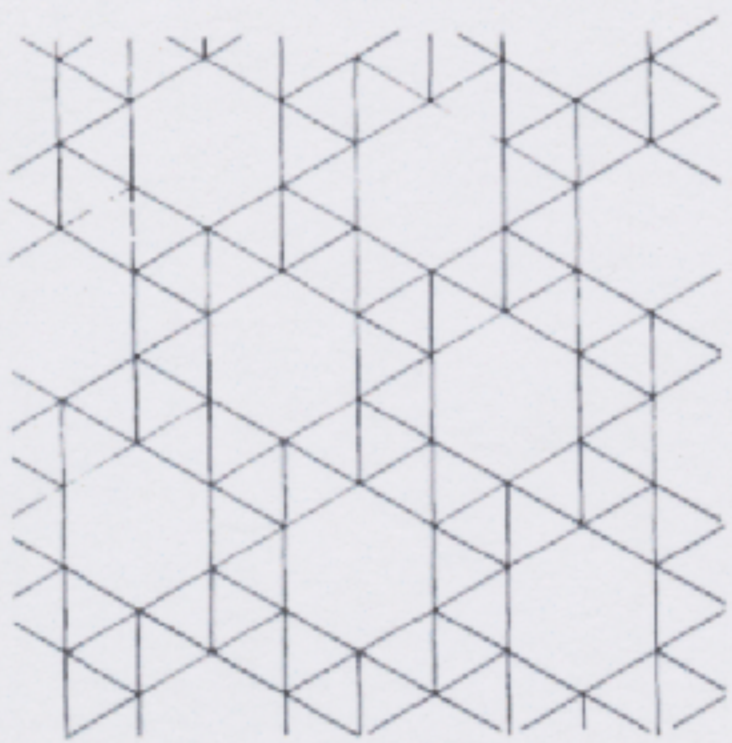


(3^6) A8458

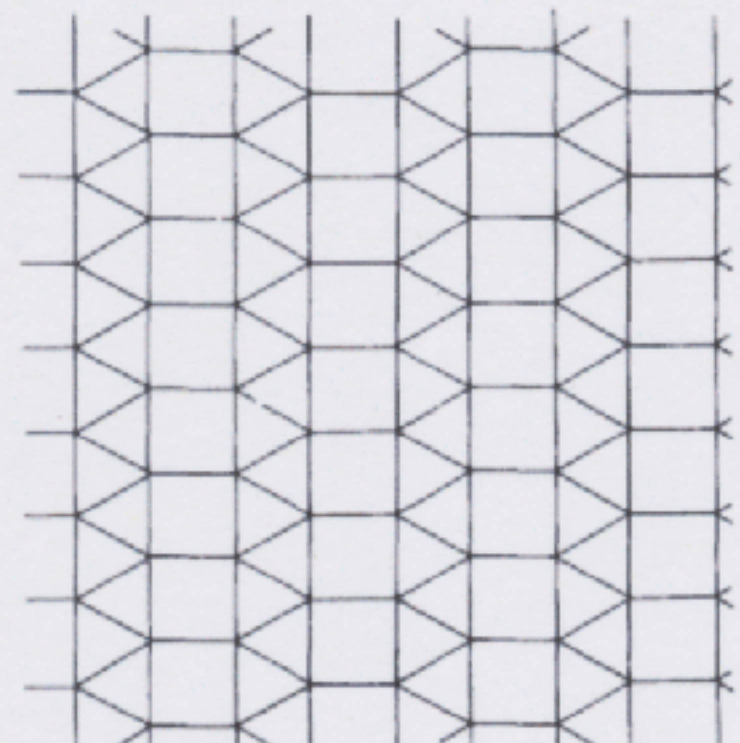


(4^4) A8574

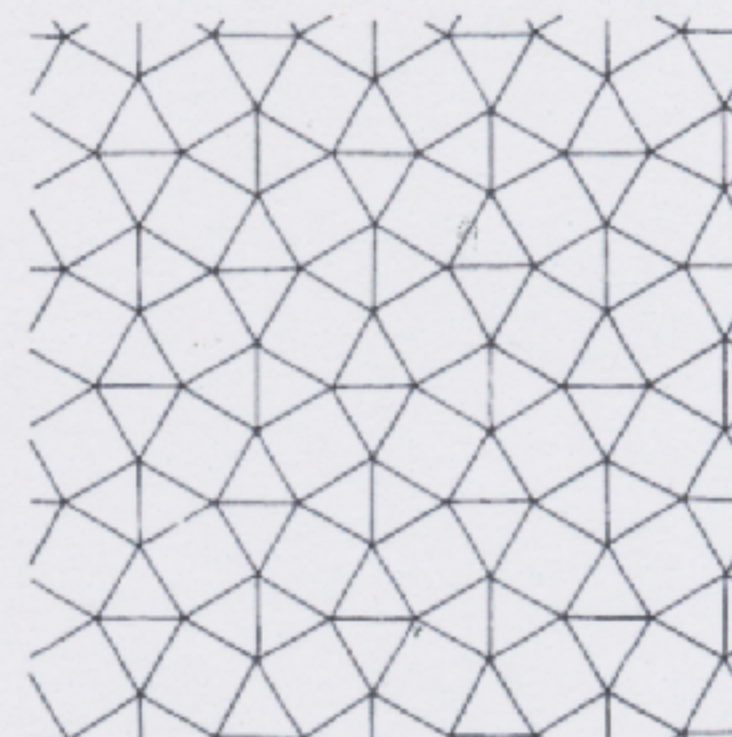
(6^3) A8486



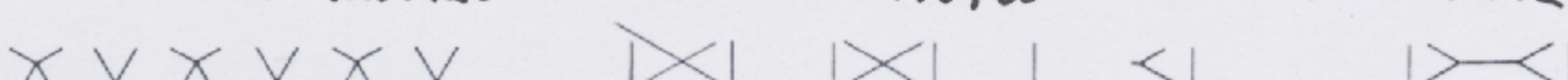
$(3^4.6)$ A250120

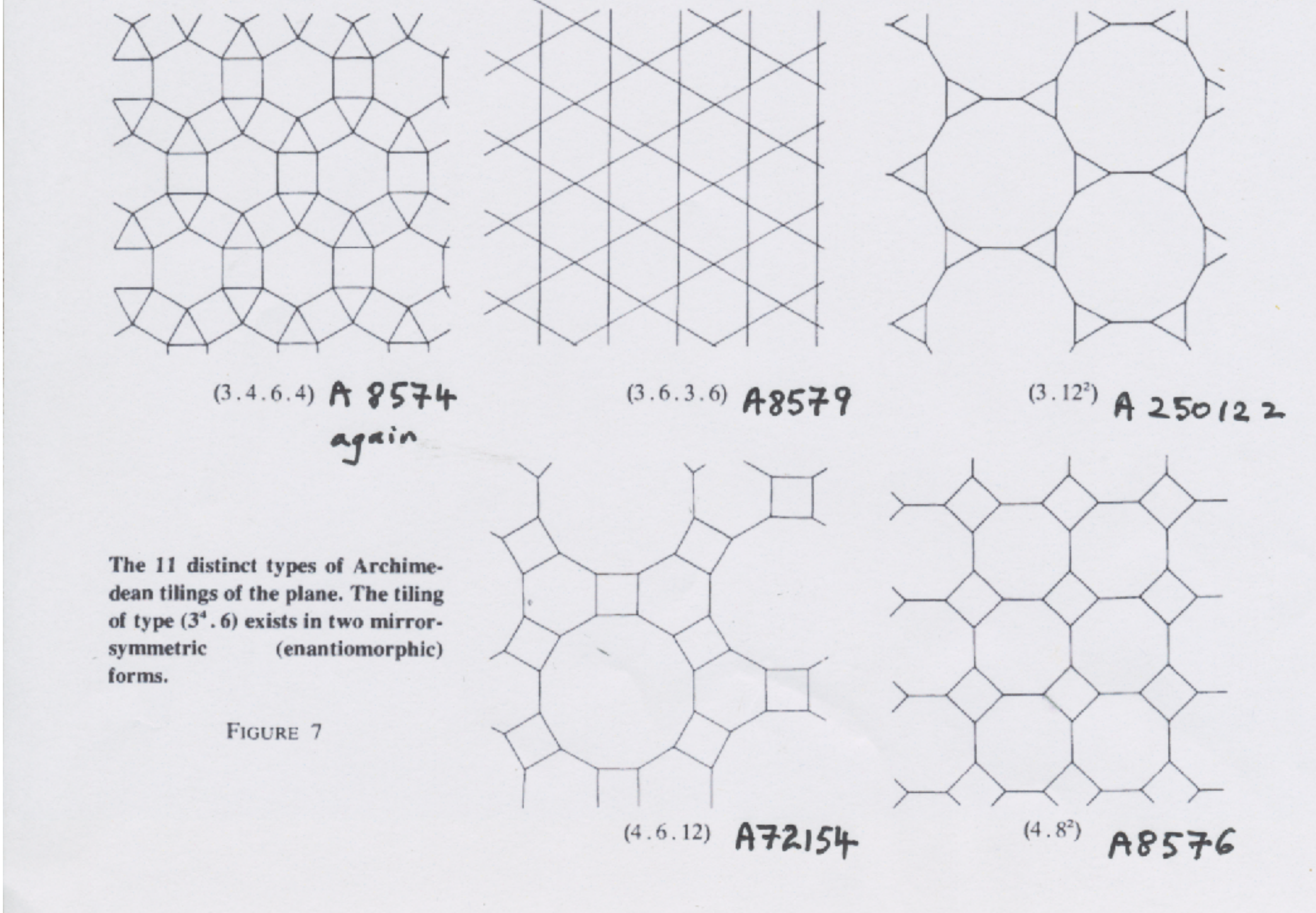


$(3^3.4^2)$ A8706



$(3^2.4.3.4)$ A219529





The 11 distinct types of Archimedean tilings of the plane. The tiling of type $(3^4.6)$ exists in two mirror-symmetric (enantiomorphic) forms.

FIGURE 7

The 11 uniform or Archimedean tilings (part 2)

Branko Grünbaum and G. C. Shephard, **Tilings** and Patterns.

The rest of the talk will be based on our new paper:

Chaim Goodman-Strauss and N. J. A. Sloane,
The Coloring Book Approach to Finding Coordination Sequences, 2018

(will soon be on arXiv and in OEIS attached to A072154 and many other entries)